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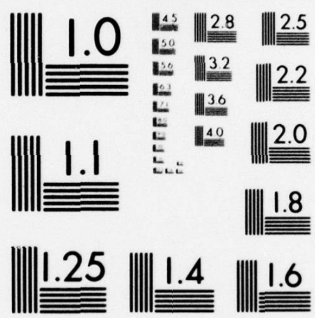

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OPERATOR VALUED FUNCTIONS AND BOUNDARY VALUE PROBLEMS FOR THE HELMHOLTZ EQUATION
I. SPHERICAL GEOMETRIES.

by

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OPERATOR VALUED FUNCTIONS AND BOUNDARY VALUE PROBLEMS FOR THE HELMHOLTZ EQUATION. I. SPHERICAL GEOMETRIES.

1. Introduction.

Let Ω be an open, connected region in $\mathbb{E}^3$ that has a smooth closed, bounded boundary $\partial\Omega$. We denote by $\overline{\Omega}$ the closure of where $\overline{\Omega} = \Omega \cup \partial\Omega$.

A number of boundary value problems of mathematical physics reduce to requiring solutions of the Helmholtz equation.

$$LW := (\nabla^2+k^2)w = f, \quad k^2 > 0 \tag{1.1}$$

defined in the region Ω and which are required to satisfy certain boundary conditions denoted typically by (bc).

It can be shown that the problem of solving the boundary value problem

$$LW = f, \quad w \in (bc) \tag{1.2}$$

can be reduced to that of solving a boundary integral equation of the form

$$(I-K)u = g \tag{1.3}$$

where K is a linear integral operator mapping $H(\partial\Omega) \equiv H$, a Hilbert space of functions defined on $\partial\Omega$, into itself, I denotes the identity operator on H whilst u and g are respectively unknown and known elements of H.

Strictly, the operator K in (1.3) depends on the parameter k appearing in (1.1) and we should in consequence regard K as an operator valued function of k. It will be our intention to make explicit this aspect of the boundary integral operator K. Specifically, we consider here the equation

$$\lambda K u = u, \quad \lambda \in \mathbb{C} \tag{1.4}$$

and in this and subsequent communications we shall concern ourselves with the following problems.

(i) The determination of $\sigma(K)$, the spectrum of K.

(ii) *The determination of the variation of* $\sigma(K)$ with k.

(iii) A discussion of the significance of the values $\lambda = \pm 1$.

2. <u>A particular case; spherical geometry</u>.

The exterior Neumann problem for the Helmholtz equation can be reduced to a study of the boundary integral equation [Kleinman and Roach, 1974]

$$(I-K)w = g \tag{2.1}$$

Here $K : L_2(\partial\Omega) \to L_2(\partial\Omega)$ is a linear operator defined by

$$(Ku)(p) = \int_{\partial\Omega} u(q) \frac{\partial\gamma}{\partial n_p}(p,q)\, dS_q \,, \quad p,q \in \partial\Omega \tag{2.2}$$

where

$$\gamma(P,Q) = -e^{ikR}/2\pi R, \quad R \equiv R(P,Q), \quad P,Q \in \Omega,$$

$$\frac{\partial}{\partial n_p} = \hat{n}_p \cdot \nabla_p$$

and $\hat{n}_p$ is the inward drawn unit vector normal to $\partial\Omega$ erected at $p \in \partial\Omega$.

We consider the equation (2.1) in the particular case when Ω is the unbounded region exterior to a sphere of radius a. Two situations present themselves; $k = 0$ and $k \neq 0$, we treat each in turn.

2(a). <u>$k = 0$, expression for eigenvalues</u>.

In this case the full form of (2.1) is

$$w(p) - \frac{\lambda}{2\pi} \int_{\partial\Omega} w(q) \frac{\partial}{\partial n_p}\left(\frac{1}{R}\right) dS_q = g(p), \quad R = R(p,q) \tag{2.3}$$

The expansion of $1/R$ in terms of complete sets or function is well known (c.f. Magnus and Oberhettinger, 1949, which hereinafter we refer to as M.O). Specifically

$$\frac{1}{R} = \sum_{n=0}^{\infty} \frac{r_<^n}{r_>^{n+1}} \cdot P_n(\cos \Theta) \tag{2.4}$$

where $\quad P := P(r_P,\theta_P,\phi_P), \quad Q := Q(r_Q,\theta_Q,\phi_Q)$

$$p := p(r_p,\theta_p,\phi_p), \quad q := q(r_q,\theta_q,\phi_q)$$

$$\cos\Theta_{pq} := \cos\Theta := \cos\theta_p \cos\theta_q + \sin\theta_p \sin\theta_q \cos(\phi_p - \phi_q) \tag{2.5}$$

$$r_< := \min(r_p, r_q), \quad r_> := \max(r_p, r_q) \tag{2.6}$$

Consequently

$$\frac{\partial}{\partial n_p}\left(\frac{1}{R}\right) = \begin{cases} \sum_{n=0}^{\infty} \frac{n r_p^{n-1}}{r_q^{n+1}} \cdot P_n(\cos\Theta), & r_p < r_q \\ -\sum_{n=0}^{\infty} (n+1) \frac{r_q^n}{r_p^{n+2}} P_n(\cos\Theta), & r_p > r_q \end{cases} \tag{2.7}$$

When $r_p = r_q = a$ [i.e. both points on the surface $\partial\Omega$] and allowing for approaching from D_-, the inside or D_+ the outside, of $\partial\Omega$, we take the mean of the two results in (2.7) and obtain

$$\frac{\partial}{\partial n_p}\left(\frac{1}{R}\right) = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{a^2} \cdot P_n(\cos\Theta)$$

to be interpreted as a distribution on $L^2(\partial\Omega)$. The homogeneous form of (2.3), in full form, can now be written.

$$w(p) - \frac{\lambda}{2\pi} \int_0^{\pi} d\theta_q \int_0^{2\pi} d\phi_q a^2 \sin\theta_q \left\{-\frac{1}{2a^2} \sum_{n=0}^{\infty} P_n(\cos\Theta)\right\} w(q) = 0$$

or

$$w(p) + \frac{\lambda}{4\pi} \sum_{n=0}^{\infty} \int_0^{\pi} d\theta_q \int_0^{2\pi} d\phi_q \sin\theta_q P_n(\cos\Theta) w(q) = 0 \tag{2.8}$$

Solutions of this equation can be found in terms of the spherical harmonics:

$$Y^m_{n,e}(\theta,\phi) := P^m_n(\cos\theta)\cos m\phi \quad , \quad e \Rightarrow \text{even in} \quad \phi$$

$$Y^m_{n,0}(\theta,\phi) := P^m_n(\cos\theta)\sin m\phi \quad , \quad 0 \Rightarrow \text{odd in} \quad \phi$$

where $P^m_n(\cos\theta)$ is the n-th associated Legendre function of order m. The system $\left\{Y^m_{n,e}, Y^m_{n,0}\right\}$ is known to be complete for $n \geq 0$, $m \leq n$ on $L_2((0,\pi)\times(0,2\pi)) = L_2(\partial\Omega)$ [c.f. M.O. and Smirnov, Vol. IV, 1964]. If in (2.8) we set:

$$w(p) = Y^m_{\substack{n,e\\0}}(\theta_p,\phi_p) \ . \tag{2.9}$$

and similarly for $w(q)$ then (2.8) becomes:

$$Y^m_{\substack{n,e\\0}}(\theta_p,\phi_p) + \frac{\lambda}{4\pi} \sum_{\ell=0}^{\infty} \int_0^{\pi} d\theta_q \int_0^{2\pi} d\phi_q \ \sin\theta_q \ P\ (\cos\Theta) Y^m_{\substack{n,e\\0}}(\theta_q,\phi_q) = 0 \ .$$

This double integral can be evaluated by the orthogonality properties of the spherical harmonics [M.O. p. 55] and we obtain

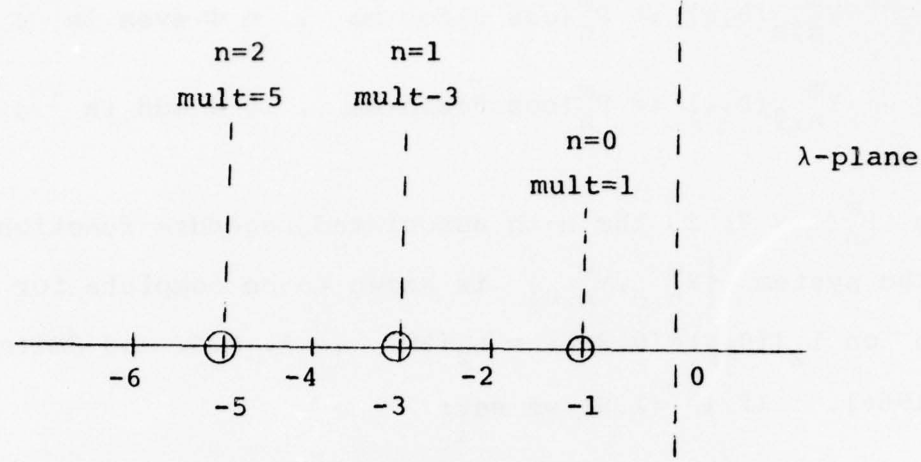
$$Y^m_{\substack{n,e\\0}}(\theta_p,\phi_p) + \frac{\lambda}{4\pi}\,\frac{4\pi}{2n+1} Y^m_{\substack{n,e\\0}}(\theta_p,\phi_p) = 0$$

$$Y^m_{\substack{n,e\\0}}(\theta_p,\phi_p)\left\{1 + \frac{\lambda}{2n+1}\right\} = 0$$

Consequently from 2.10 we see that the eigenvalues of (2.3) are the scalars

$$\lambda_n = -(2n+1) \tag{2.11}$$

and λ_n is an eigenvalue of (2.3) of multiplicity $2n+1$ when $n \geq 0$. This follows because for fixed n we have n eigenfunctions Y^m_n, $0 \leq m \leq n$ for the even situation and $1 \leq m \leq n$ for the odd; hence multiplicity $2n+1$.



Eigenvalue distribution: $\lambda_n(k)\Big|_{k=0}$

2. (b) <u>$k \neq 0$, expression for eigenvalues.</u>

In the case $k \neq 0$ the homogeneous form of equation (2.1) becomes

$$w(p) - \frac{\lambda}{2\pi}\int_{\partial\Omega} w(q) \frac{\partial}{\partial n_p}\left(\frac{e^{ikR}}{R}\right) dS_q = 0. \tag{2.12}$$

Using known expansions [M.O. p. 21] we have for this case

$$\frac{e^{ikR}}{R} = ik \sum_{n=0}^{\infty} (2n+1) j_n(kr_<) h_n^{(1)}(kr_>) p_n(\cos\Theta) \tag{2.13}$$

where

$$z_n(x) = \sqrt{\frac{\pi}{2x}}\; Z_{n+1/2}(x) \tag{2.14}$$

and z_n represents either j_n the spherical Bessel function or $h_n^{(1)}$ the spherical Hankel function of the first kind while Z represents the cylindrical counterparts.

Proceeding as before we differentiate (2.13) with respect to n_p, obtain expressions for p tending to $\partial\Omega$ from inside and outside $\partial\Omega$ then add and finally divide by 2 to obtain:

$$\frac{\partial}{\partial n_p}\left(\frac{e^{ikR}}{R}\right) = \frac{ik^2}{2}\sum_{n=0}^{\infty}(2n+1)\left\{j_n(ka)h_n^{(1)\prime}(ka)+j_n'(ka)h_n^{(1)}(ka)\right\}P_n(\cos\Theta) \tag{2.15}$$

Substituting (2.15) into (2.12) we obtain:

$$w(p)-\frac{\lambda}{2\pi}\int_0^{\pi}d\theta_q\int_0^{2\pi}d\phi_q w(q)a^2\sin\theta_q\,\frac{ik^2}{2}\sum_{n=0}^{\infty}(2n+1)\left\{j_n(ka)h_n^{(1)\prime}(ka)+\right.$$

$$\left.+\,j_n'(ka)h_n^{(1)}(ka)\right\}P_n(\cos\Theta) = 0.$$

If we now interchange the order of integration and abbreviate the integrand in an obvious way then we obtain:

$$w(p)-\frac{\lambda i(ka)^2}{4\pi}\sum_{n=0}^{\infty}(2n+1)\left\{j_nh_n^{(1)\prime}+j_n'h_n^{(1)}\right\}\int_0^{\pi}d\theta_q\int_0^{2\pi}d\phi_q\sin\theta_q P_n(\cos\Theta)w(q)$$

$$= 0. \tag{2.16}$$

As before set

$$w(p) = Y^m_{n,\overset{e}{o}}(\theta_p,\phi_p) \tag{2.17}$$

and employ the associated orthogonality properties [M.O. p. 55] to obtain

$$Y^m_{n,\overset{e}{o}}(\theta_p,\phi_p)-\frac{\lambda i}{4\pi}(ka)^2(2n+1)\left\{j_nh_n^{(1)\prime}+j_n'h_n^{(1)}\right\}\left\{\frac{4}{2n+1}\cdot Y^m_{n,\overset{e}{o}}\right\} = 0.$$

This implies that

$$Y_{n,\,\substack{e\\o}}^{m}(\theta_p,\phi_p)\left\{1-\lambda_i(ka)^2[j_n h_n^{(1)\prime}+h_n^{(1)}j_n']\right\}=0$$

and consequently we deduce that

$$\lambda_n = \frac{1}{i(ka)^2[j_n' h_n^{(1)}+j_n h_n^{(1)\prime}]} \tag{2.18}$$

which is an eigenvalue of (2.12) of multiplicity $2n+1(n\geq 0)$. Now it can be shown [Abramowitz and Stegun, 1964] that the Wronskian is given by

$$W(j_n,h_n^{(1)}):=J_n h_n^{(1)\prime} - j_n' h_n^{(1)} = \frac{i}{(ka)^2}\ .$$

Hence (2.18) can be rewritten in the form:

$$\lambda_n = \frac{1}{i(ka)^2}\left\{\frac{1}{i/(ka)^2+2j_n' h_n^{(1)}}\right\}$$

or

$$\lambda_n = \frac{1}{-1+2i(ka)^2 j_n' h_n^{(1)}} \tag{2.19}$$

where it is to be understood that $j_n' h_n^{(1)} \equiv j_n'(ka)h_n^{(1)}(ka)$ and ' denotes differention with respect to the argument. Here the eigenvalues λ_n are dependent on the value of k. Consequently we would expect that as $k \to 0$ we would recover the eigenvalue spectrum obtained in the previous section. That this is indeed the case can be seen by means of the following asymptotic analysis. From [M.O. p. 22) we have the expansions:

$$h_n^{(1)}(z) = i^{-n-1}\,\frac{e^{iz}}{z}\sum_{m=0}^{n}(-1)^m\,\frac{(n+1/2;m)}{(2iz)^m} \tag{2.20}$$

$$j_n(z) = \frac{\sqrt{\pi}}{2}\cdot\left(\frac{z}{2}\right)^{n}\sum_{m=0}^{\infty}\frac{(-1)^m(z/2)^{2m}}{m!\,\Gamma(n+m+3/2)} \tag{2.21}$$

where

$$(n+1/2,m) = \frac{(-1)^m \ (n+1)_m (-n)_m}{m!}$$

and

$$(\nu)_m := \frac{\Gamma(\nu+m)}{\Gamma(\nu)}$$

consequently (2.20) can be written in asymptotic form (for small z) as

$$h_n^{(1)}(z) = i^{-n-1} \frac{e^{iz}}{z}(-1)^n \frac{(n+1/2,n)}{(2iz)^n} [1+0(z)]$$

$$= \frac{1}{i} \frac{e^{iz}}{2^n z^{n+1}}(n+1/2,n) [1+0(z)] \tag{2.22}$$

and we can deduce that

$$h_n^{(1)'}(z) = \frac{-(n+1)(n+1/2,n)e^{iz}}{i2^n z^{n+2}} (1+0(z)) \ . \tag{2.23}$$

Similarly we obtain

$$j_n(z) = \frac{\sqrt{\pi}}{2} \left(\frac{z}{2}\right)^n \frac{1}{\Gamma(n+3/2)} + 0(z^{n+2}) \tag{2.24}$$

and

$$j_n'(z) = \frac{\sqrt{\pi}}{2^{n+1}} \cdot \frac{nz^{n-1}}{\Gamma(n+3/2)} + 0(z^{n+1}). \tag{2.25}$$

Consequently

$$j_n'(z)h_n^{(1)}(z) = \left\{\frac{n\sqrt{\pi}}{2^{n+1}} \cdot \frac{z^{n-1}}{\Gamma(n+3/2)} + 0(z^{n+1})\right\}\left\{\frac{e^{iz}(n+1/2,n)}{i2^n z^{n+1}}[1+0(z)]\right\}$$

$$= [a_n z^{n-1}+0(z^{n+1})]\left\{b_n e^{iz}/z^{n+1}[1+0(z)]\right\}$$

$$= a_n b_n \frac{e^{iz}}{z^2} [1+0(z)] \tag{2.26}$$

where a_n and b_n are defined in an obvious manner.

Setting

$$c_n := a_n b_n = \frac{n\sqrt{\pi}(n+1/2,n)}{2^{n+1}\Gamma(n+3/2)i2^n} = \frac{1}{i}\,\frac{n\sqrt{\pi}(n+1/2,n)}{2^{2n+1}\Gamma(n+3/2)} \tag{2.27}$$

and

$$z = ka\ ,$$

we find on using (2.26),(2.27) that (2.19) can be written in the form

$$\lambda_n = \frac{1}{-1+2iz^2\left\{c_n\frac{e^{iz}}{z^2}[1+0(z)]\right\}} \tag{2.28}$$

$$\lambda_n = \frac{1}{-1+2ic_n e^{iz}+0(z)}\ . \tag{2.29}$$

In order that this result (2.29) should agree with (2.11) we must have

$$2ic_n = \frac{2n}{2n+1}\ . \tag{2.30}$$

That this is indeed the case can be seen as follows. From the definition of (n+1/2,m) and (2.27) we obtain

$$2ic_n = \frac{n\sqrt{\pi}(n+1/2,m)}{2^{2n}\Gamma(n+3/2)} = \frac{n\Gamma(1/2)(-1)^n(n+1)_n(-n)_n}{2^{2n}\ n!\Gamma(n+3/2)}$$

and since

$$(-1)^n(n+1)_n(-n)_n = \Gamma(2n+1) = 2^{2n}n!\,\frac{\Gamma(n+1/2)}{\Gamma(1/2)}$$

we find that

$$2ic_n = \frac{n\Gamma(1/2)\Gamma(n+1/2)}{\Gamma(1/2)\Gamma(n+3/2)} = \frac{n}{n+1/2}$$

as required.

For large values of z it follows from (2.20) and the fact that $(n+1/2,0) = 1$, that for the spherical Hankel function of the first kind,

$$h_n^{(1)}(z) = i^{-n-1}\frac{e^{iz}}{z}(1+0(1/z))$$

and

$$h_n^{(1)}(z) = i^{-n}\frac{e^{iz}}{z}(1+0(1/z)).$$

Similarly, for the spherical Hankel function of the second kind we have that

$$h_n^{(2)}(z) = i^{(n+1)}\frac{e^{-iz}}{z}(1+0(1/z)).$$

Hence

$$\begin{aligned} j_n(z) &:= \frac{1}{2}\left(h_n^{(1)}(z)+h_n^{(2)}(z)\right) \\ &= \frac{1}{2z}\left(i^{-n-1}e^{iz}+i^{n+1}e^{-iz}\right)(1+0(1/z)) \end{aligned}$$

and

$$j_n(z)h_n^{(1)'}(z) = \frac{1}{2z^2}\left(i^{-2n+1}e^{2iz}+i\right)(1+0(1/z)).$$

Therefore, from (21.9) and the definition of the Wronskian it follows that

$$\begin{aligned} \lambda_n &= \frac{1}{1+2iz^2 j_n h_n^{(1)'}} \\ &= \left\{1+\left[(-1)^n e^{2iz}-1\right]\left[1+0(1/z)\right]\right\}^{-1}. \end{aligned}$$

Consequently, the asymptotic form of the eigenvalues for large values of z is

$$\lambda_n = (-1)^n e^{-2iz}[1+0(1/z)]. \tag{2.31}$$

The results of numerical calculations of these eigenvalues and their reciprocals are plotted in Appendix II. The first eleven eigenvalues (equ. 2.18) are shown as functions of ka illustrating how the asymptotic behavior (equs. 2.29 and 2.31 for low and high frequencies respectively) is approached.

3. On the polar decomposition of K.

The results of the previous section indicate that as k varies the eigenvalues of K become complex. Furthermore, as k increases the modulus of the eigenvalues tends to unity. This suggests that K should be decomposable either as the sum or product of operators, one of which is unitary, in order to be able to exploit the behaviour of the eigenvalues of K for large k.

One way of effecting such a decomposition is afforded in the so-called polar decomposition of operators [Naylor and Sell, 1971]. Specifically, given a linear, normal operator $K: H \to H$, where H is a separable Hilbert space then there exists a unique decomposition of K in the form

$$K = UR = RU \tag{3.1}$$

where $R: H \to H$ is a positive, self-adjoint, linear operator and $U: H \to H$ is a unitary operator.

Now the boundary integral equations in which we are interested have the typical form

$$(I-\lambda K)w = g \in H, \ \lambda \in \mathbb{C} . \tag{3.2}$$

Therefore, if K is normal then (3.2) can be rewritten either in the form

$$(U^{-1}-\lambda R)w = \tilde{g} \ , \quad g = U\tilde{g} \tag{3.3}$$

or as

$$(R^{-1}-\lambda U)w = \tilde{g} \ , \quad g = R\tilde{g} \tag{3.4}$$

Consequently in the light of the results of the previous sections we are led to make the following conjectures.

K1. As $z := ka \to \infty$ then $K \to U$ therefore $R \to \pm I$ and $R^{-1} \to \pm I$ where I is the identity on H. Consequently, the high frequency behavior of the required solutions to (3.2) is determined from the solutions of the equation

$$(I-\lambda U)w_{\infty} = g_{\infty}, \quad w_{\infty} = \lim_{z\to\infty} Rw = \pm\lim_{z\to\infty} w, \quad g_{\infty} = \lim_{k\to\infty} g \tag{3.5}$$

where this equation has been obtained from (3.2) and (3.4) in the limit as $z \to \infty$.

K2. As $z: ka \to 0$, K becomes self adjoint therefore $U \to \pm I$ and $U^{-1} \to \pm I$. Consequently, the low frequency behavior of the required solution to (3.2) is determined from the solutions of the equation

$$(I-\lambda R)w_o = g_o, \quad w_o = \lim_{z\to 0} Uw = \lim_{z\to 0} w, \quad g_o = \lim_{k\to 0} g \tag{3.6}$$

where the equation has been obtained from (3.4) in the limit as $z \to 0$.

The Spectral Theorem (Naylor and Sell, 1971) states that if $K:H \to H$ is a compact, normal operator then there exists a resolution of the identity $\{E_n\}$ and a sequence of complex numbers $\{\mu_n\}$, such that

$$K = \sum_n \mu_n E_n \tag{3.7}$$

where the convergence in (3.7) is understood to be in terms of the uniform operator norm topology.

$\{E_n\}$ is a resolution of the identity provided

(i) E_n are orthogonal projections, i.e., $E_n^2 = E_n = E_n^*$ and $R(E_n) \perp N(E_n)$

(ii) $E_n E_m = 0$, $n \neq m$

(iii) $\sum_n E_n = I$.

In the case under consideration, μ_n and E_n may be determined from 2.12, 2.15, 2.18 to be

$$\mu_n = \lambda_n^{-1} = iz^2\left\{j_n'(z)h_n^{(1)}(z)+j_n(z)h_n^{(1)\prime}(z)\right\} \tag{3.8}$$

$$E_n(\cdot) = \frac{2n+1}{4\pi}\int_0^{\pi} d\theta_q \int_0^{2\pi} d\phi_q \sin\theta_q P_n(\cos\Theta_{p,q})(\cdot). \tag{3.9}$$

The well-known properties of the spherical harmonics (M.O. p. 55) enable one to verify that (i)-(iii) are fulfilled.

In general, provided $|\mu_n| \neq 0$, and since the E_n are orthogonal projections, we can write

$$K = \sum_n \mu_n E_n = \sum_n |\mu_n| E_n \sum_m \mu_m |\mu_m|^{-1} E_m = RU . \tag{3.10}$$

Now, any operator K which can be written in the form (3.7) is called a weighted sum of projections. Therefore the operators R and U appearing in (3.8) and defined by

$$R := \sum_n |\mu_n| E_n , \quad U := \sum_n \mu_n |\mu_n|^{-1} E_n \tag{3.11}$$

are both weighted sums of projections and R is self-adjoint since the weights are real ($|\mu_n| \in R$). Furthermore, by first recalling the properties of the resolution of the identity, elementary manipulation shows that U is unitary. Therefore (3.10) provides the required polar decomposition of K. We are now in a position to examine the conjectures K1 and K2 if we recall the asymptotic estimates obtained in the previous sections, namely

$$\lambda_n(z) = \begin{cases} -(2n+1) + 0(z) & \text{, for small } z \text{ and fixed } n. \\ (-1)^n e^{-2iz} + 0(1/z) & \text{, for small } z \text{ and fixed } n. \end{cases}$$

Consequently

$$\mu_n(z) := \lambda_n^{-1}(z) = \begin{cases} \frac{-1}{(2n+1)}[1+0(z)] & \text{for small } z \text{ and fixed } n, \\ (-1)^n e^{2iz}[1+0(1/z)] & \text{for large } z \text{ and fixed } n, \end{cases} \tag{3.12}$$

and we obtain by direct substitution of these quantities into (3.11

$$R = \sum_{n=0}^{\infty} |\mu_n| E_n = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi a^2} \int_{\partial\Omega} P_n(\cos \Theta_{pq})\ (\cdot) dS_q [1+0(z)]$$

$$= -I(1+0(1/z)) \quad \text{for large } z\ . \tag{3.13}$$

$$U = \sum \mu_n |\mu_n|^{-1} E_n = - \sum_{n=0}^{\infty} \frac{(2n+1)}{4\pi a^2} \int_{\partial\Omega} P_n(\cos \Theta_{pq})(\cdot) dS_q [1+0(z)]$$

$$= -I(1+0(z)) \quad \text{for small } z\ . \tag{3.14}$$

4. Conclusion,

In this note we have begun an investigation into both the theoretical and practical consequences of treating certain commonly occuring operators of mathematical physics as operator valued functions of a parameter.

As a first consideration of this problem we have confined our attention to boundary value problems associated with the

Helmholtz equation in the particular case of spherical symmetry. We have seen that when $\partial\Omega$ is a sphere then equation (1.4) can be solved analytically. Consequently a number of quantities which are known to depend upon a certain parameter can be computed directly and so the dependence can be exhibited. Specifically, we have seen that as k varies, the eigenvalues of K become complex. Furthermore we have seen that as k increases the modulus of each eigenvalue tends to unity. This suggests that K should be decomposable as the product of operators, one of which is unitary, in order to exploit the behavior of the eigenvalues of K for large k. This decomposition was explicitly exhibited for the spherical boundary. Promising as this appears at first sight, it turns out not to be a very practical proposition in general, since its implementation requires an intimate knowledge of all the eigenvalues and eigenvectors of K. Furthermore such polar decomposition of K is only conveniently available for compact normal operators and while it can be shown (Appendix I) that K is normal for spherical geometries, this may not be true in general. Nevertheless we shall show in subsequent communications that there are factorization methods which do offer good practical prospects especially if instead of K attempts are made to factorize $(I-K)^{-1}$, the resolvent of K.

Finally we remark that the equation whose solution is sought, (2.1), has obvious difficulties for those values of ka for which $\lambda_n(ka)=1$, where $\lambda_n(ka)$ are the eigenvalues of equ. 1.4. To show that such critical values do occur in the spherical example treated here, the quantity $|\lambda_n(ka)-1|$ is plotted as functions of ka for $0 \leq n \leq 10$ in Appendix II.

Appendix I. Concerning the normality of boundary integral operators.

On $L_2(\partial\Omega)$ we define the inner product

$$(u,v) = \int_{\partial\Omega} u(p)\overline{v(p)}dS_p \quad , \quad p \in \partial\,\Omega \tag{A1.1}$$

where dS_p denotes an element of area of the surface $\partial\Omega$ taken with respect to the coordinates of a point $p \in \partial\,\Omega$.

We introduce an integral operator $K: L_2(\partial\Omega) \to L_2(\partial\Omega)$ defined by

$$(Ku)(p) := \int_{\partial\Omega_q} \frac{\partial\gamma(p,q)}{\partial n_p} \, u(q)dS_q \; , \quad p,q \in \partial\,\Omega \tag{A1.2}$$

where the kernel of this operator is generated by $\gamma(P,Q)$, $P, Q \in \overline{\Omega}$, a function of position in $\overline{\Omega} = \Omega \cup \partial\Omega$. The domain functional is conveniently taken to be the fundamental solution of the associated partial differential equation. In this example, in which we are considering the Helmholtz equation, we take

$$\gamma(P,Q) = -\frac{e^{ikR}}{2\pi R} \quad , \quad R := |\vec{PQ}| \quad , \quad P,Q \in \overline{\Omega} \, . \tag{A1.3}$$

The adjoint of K is obtained by considering

$$\begin{aligned}(Ku,v) &= \int_{\partial\Omega_p} (Ku)(p)\overline{v(p)}dS_p \\ &= \int_{\partial\Omega_p}\left\{\int_{\partial\Omega_q} \frac{\partial\gamma}{\partial n_p}(p,q)u(q)dS_q\right\}\overline{v}(p)dS_p \\ &= \int_{\partial\Omega_q} u(q)\left\{\int_{\partial\Omega_p} \frac{\partial\gamma}{\partial n_p}(p,q)\overline{v}(p)dS_p\right\}dS_q \\ &= (u,K^*v)\end{aligned}$$

where

$$(K^*v)(q) := \int_{\partial\Omega_p} \frac{\partial\overline{\gamma}}{\partial n_p}(p,q)v(p)dS_p . \qquad \text{(A1.4)}$$

In order to investigate the normality of the operator K we examine the properties of

$$S := K^*K - KK^* \qquad \text{(A1.5)}$$

Substituting for K and K^* in A1-5 we obtain

$$(Su)(p) = \int_{\partial\Omega_r}\int_{\partial\Omega_q} u(q)\left\{\frac{\partial\gamma}{\partial n_r}(r,q)\frac{\partial\overline{\gamma}}{\partial n_r}(r,p) - \frac{\partial\gamma}{\partial n_p}(p,r)\frac{\partial\overline{\gamma}}{\partial n_q}(q,r)\right\}dS_q dS_r$$

(A1.6)

Consequently

$$D(p,q) := \int_{\partial\Omega_r}\left\{\frac{\partial\gamma}{\partial n_r}(r,q)\frac{\partial\overline{\gamma}}{\partial n_r}(r,p) - \frac{\partial\gamma}{\partial n_p}(p,r)\frac{\partial\overline{\gamma}}{\partial n_q}(q,r)\right\}dS_r = 0 \qquad \text{(A1.7)}$$

is a necessary and sufficient condition for the normality of K.

To see that A1.7 is satisfied for the Helmholtz equation and $\partial\Omega$ a sphere we first write (2.15) in the form

$$\frac{\partial\gamma}{\partial n_p}(p,q) = \sum_{n=0}^{\infty} F_n(a)P_n(\cos\Theta_{pq}) := \sum_{n=0}^{\infty} F_n(a)P_n(p,q) \qquad \text{(A1.7)}$$

where $F_n(a) = \frac{ik^2}{2}(2n+1)[j_n(ka)h_n^{(1)\prime}(ka) + j_n'(ka)h_n^{(1)}(ka)]$.

Substituting A1.7 into A1.6 yields

$$D(p,q) = \int_{\partial\Omega_r}\left\{\sum_{n=0}^{\infty} F_n(a)P_n(r,q)\sum_{m=0}^{\infty}\overline{F_m}(a)P_m(r,p)\right.$$

$$\left. - \sum_{n=0}^{\infty} F_n(a)P_n(p,r)\sum_{m=0}^{\infty}\overline{F_m}(a)P_m(q,r)\right\}dS_r$$

$$= \sum_{n=0}^{\infty} \sum_{m=0} F_n(a)\overline{F_m(a)} \int_{\partial\Omega_r} \left\{ P_n(r,q)P_m(r,p) - P_n(p,r)P_m(q,r) \right\} dS_r$$

$$= \sum_{n=0}^{\infty} F_n(a)\overline{F_n}(a) \frac{4\pi}{2n+1} \left\{ P_n(p,q) - P_n(p,q) \right\}$$

$$= 0.$$

Hence for the particular case of the Helmholtz equation and a spherical surface the boundary integral operator $K: L_2(\partial\Omega) \to L_2(\partial\Omega)$ is normal.

Appendix II. Numerical results.

Equation (2.19) defines λ_n explicitly as

$$\lambda_n = \frac{1}{-1+2i(ka)^2 j_n'(ka)h_n^{(1)}(ka)} .$$

To illustrate the behavior of the eigenvalues as functions of ka, λ_n is plotted as a function of ka for $0 \leq n \leq 10$ and $0 \leq ka \leq 10$ in Figures 1-11. Notice that the asymptotic behavior for small and large ka is evident although for the larger values of n, values of ka greater than 10 are required before the asymptotic approximation is reasonable. In Figures 1-6, 8, 10 and 11, the horizontal and vertical scales are different so extra care is urged in interpreting these graphs. The reciprocal eigenvalues $\mu_n = \frac{1}{\lambda_n}$ are plotted in Figs. 12-22. The advantage in plotting in this way is that the μ_n are bounded and the same scale can be employed for all.

Since the equation to be solved is

$$(I-\lambda k)w = f$$

with $\lambda = 1$, it is of interest to observe how close this value of 1 is to an eigenvalue. To exhibit this the quantity $|\lambda_n(ka)-1|$ is plotted as a function of ka for $0 \leq n \leq 10$ and $0 \leq ka \leq 10$ in Figures 23-33.

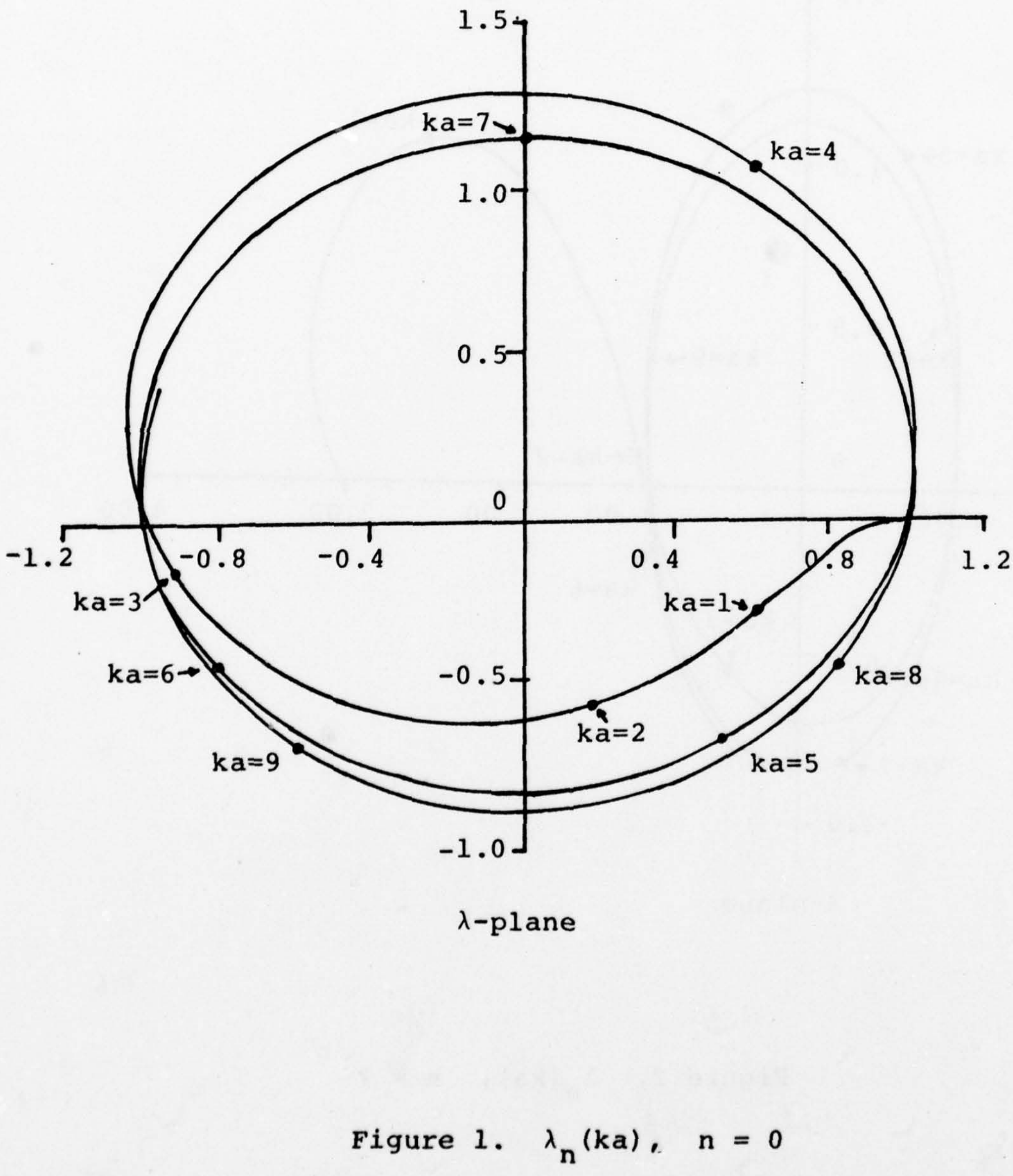


Figure 1. $\lambda_n(ka)$, $n = 0$

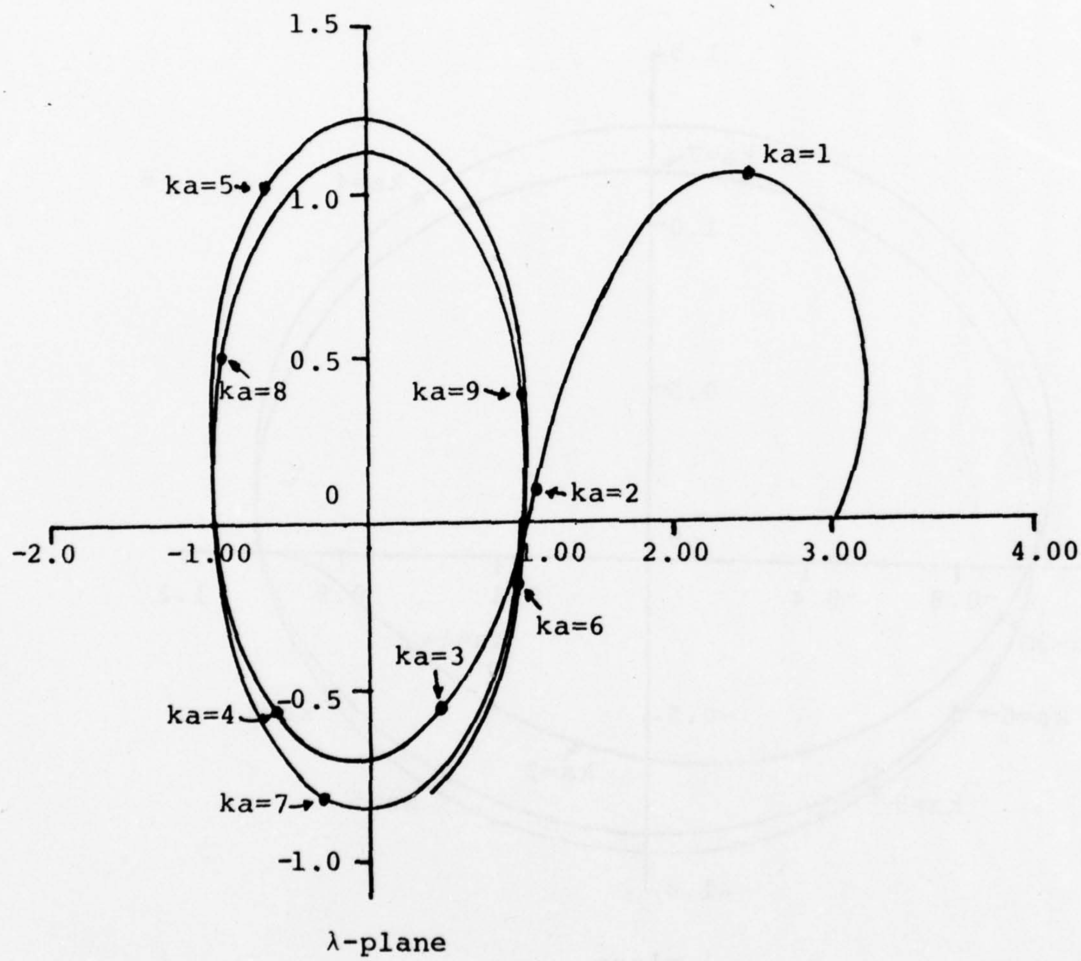


Figure 2. $\lambda_n(ka)$, $n = 1$

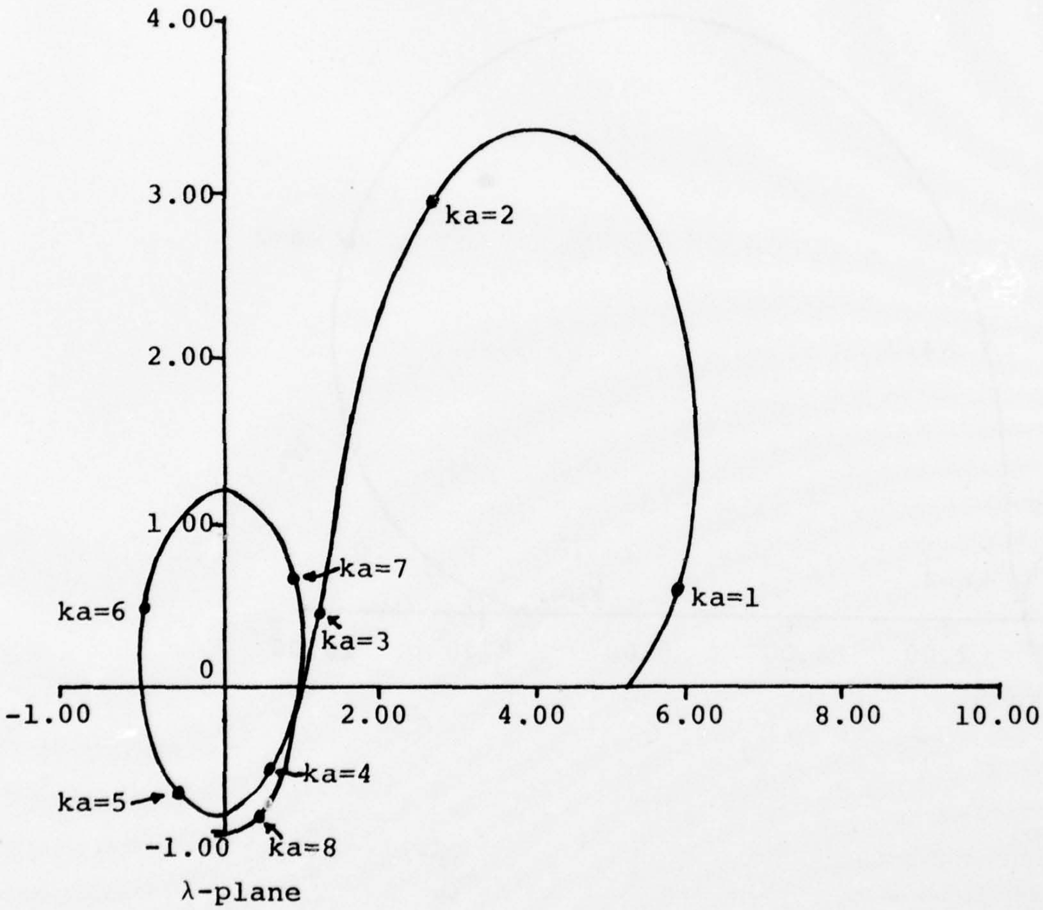


Figure 3. $\lambda_n(ka)$, $n = 2$

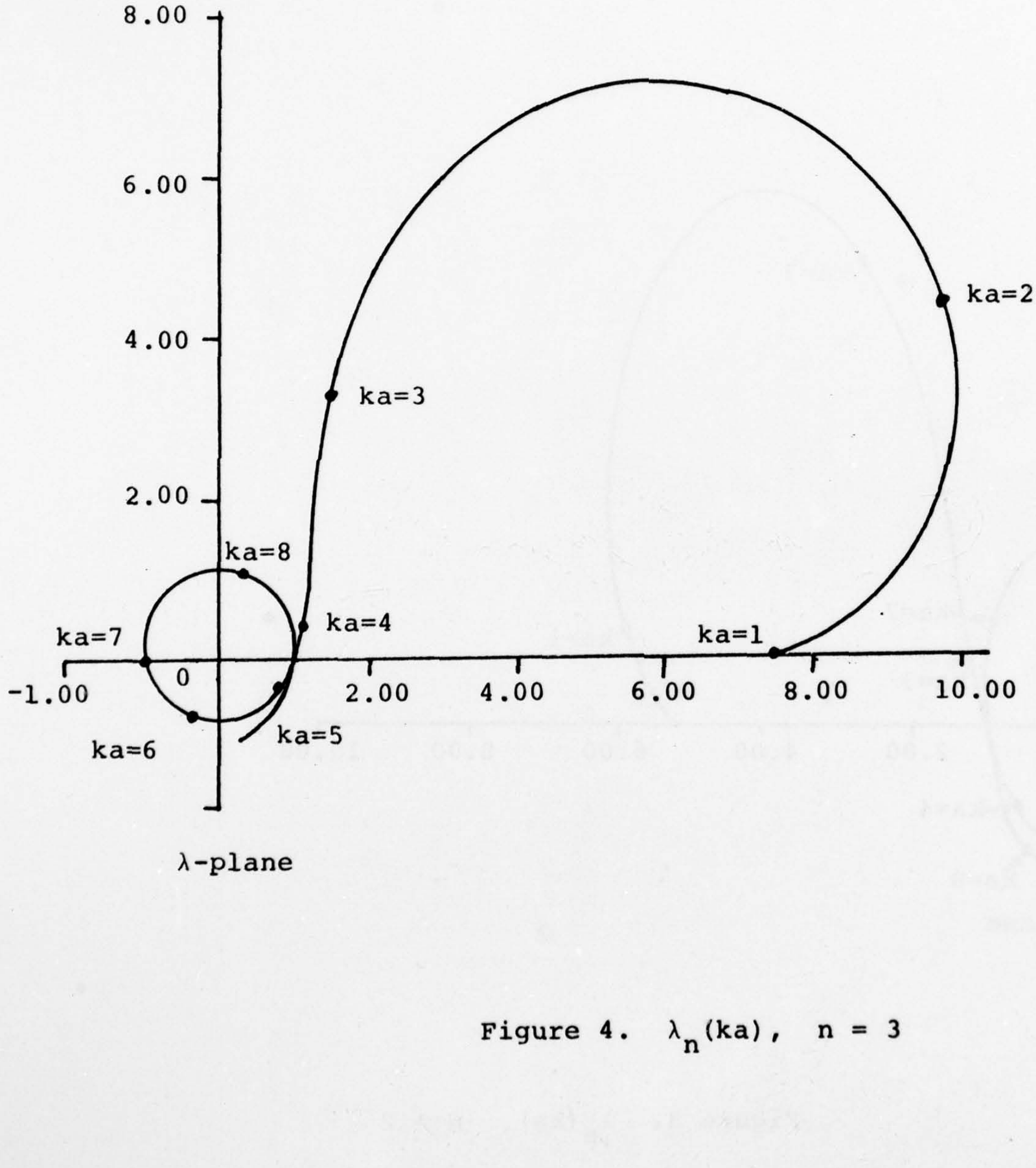


Figure 4. $\lambda_n(ka)$, $n = 3$

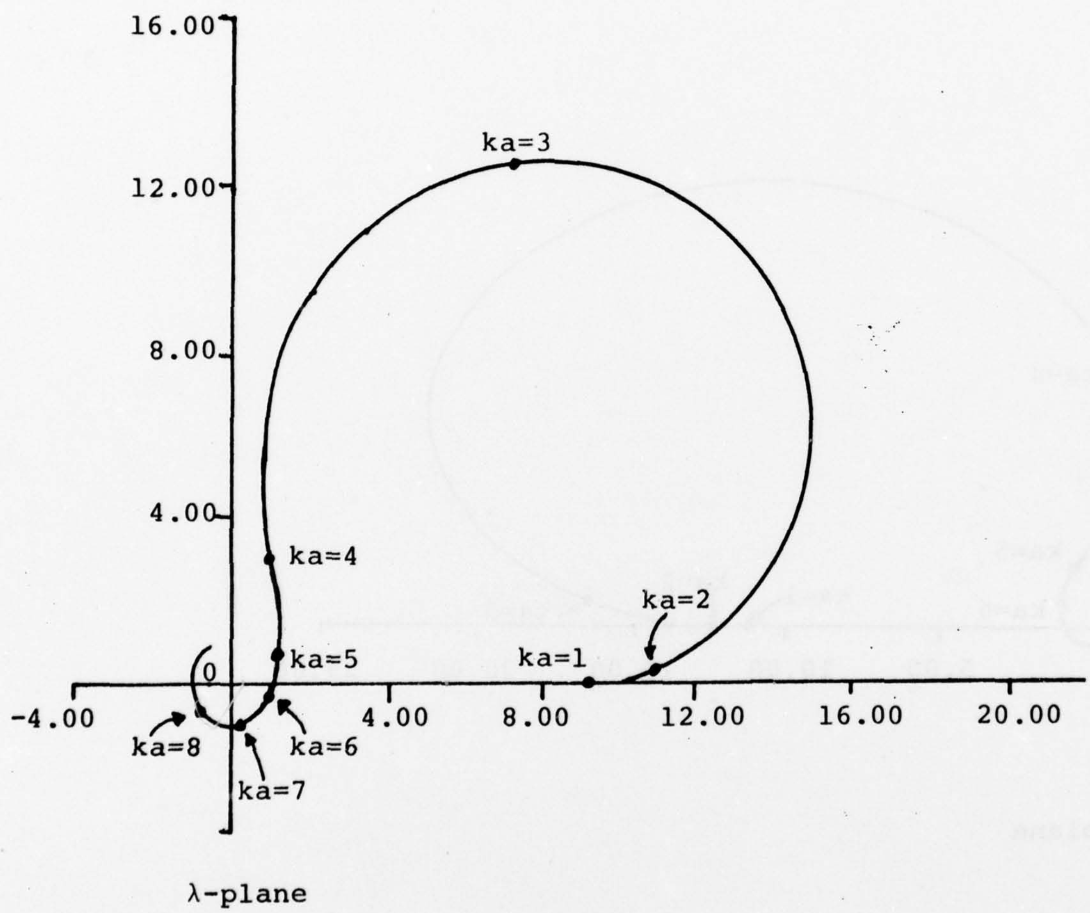


Figure 5. $\lambda_n(ka)$, n = 4

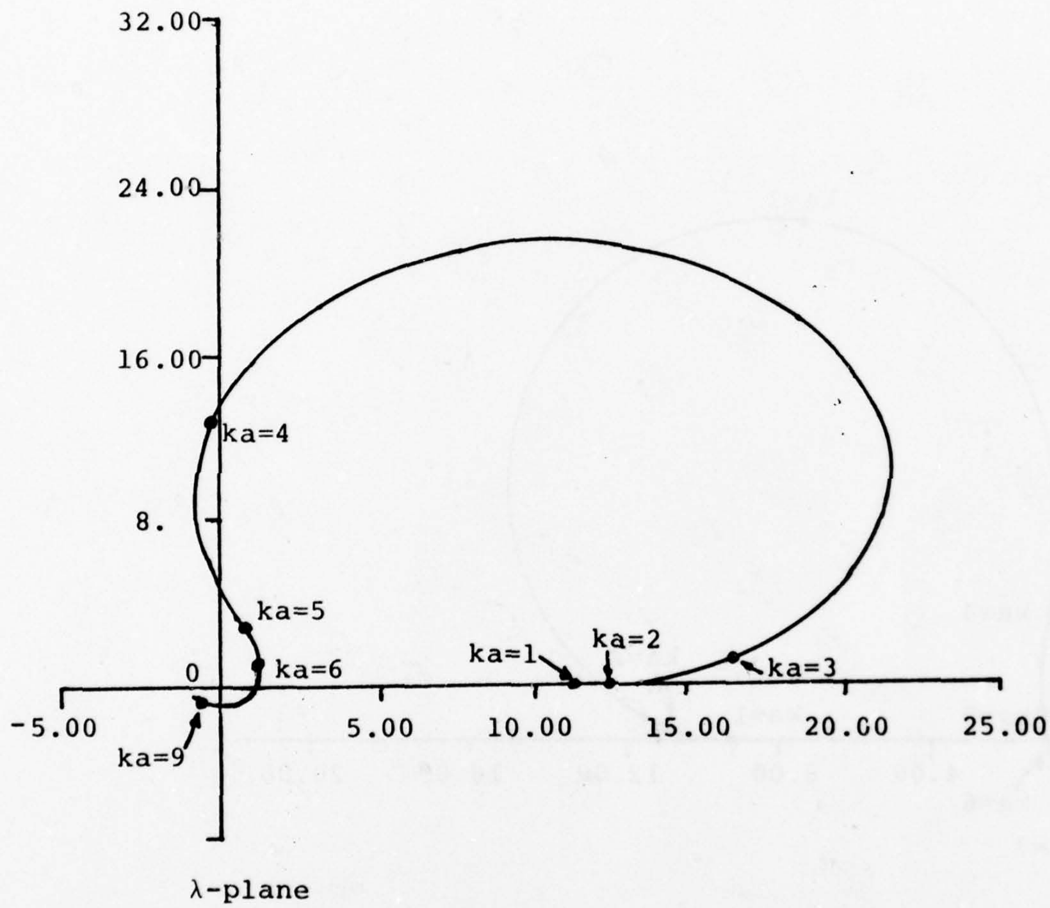


Figure 6. $\lambda_n(ka)$, $n = 5$

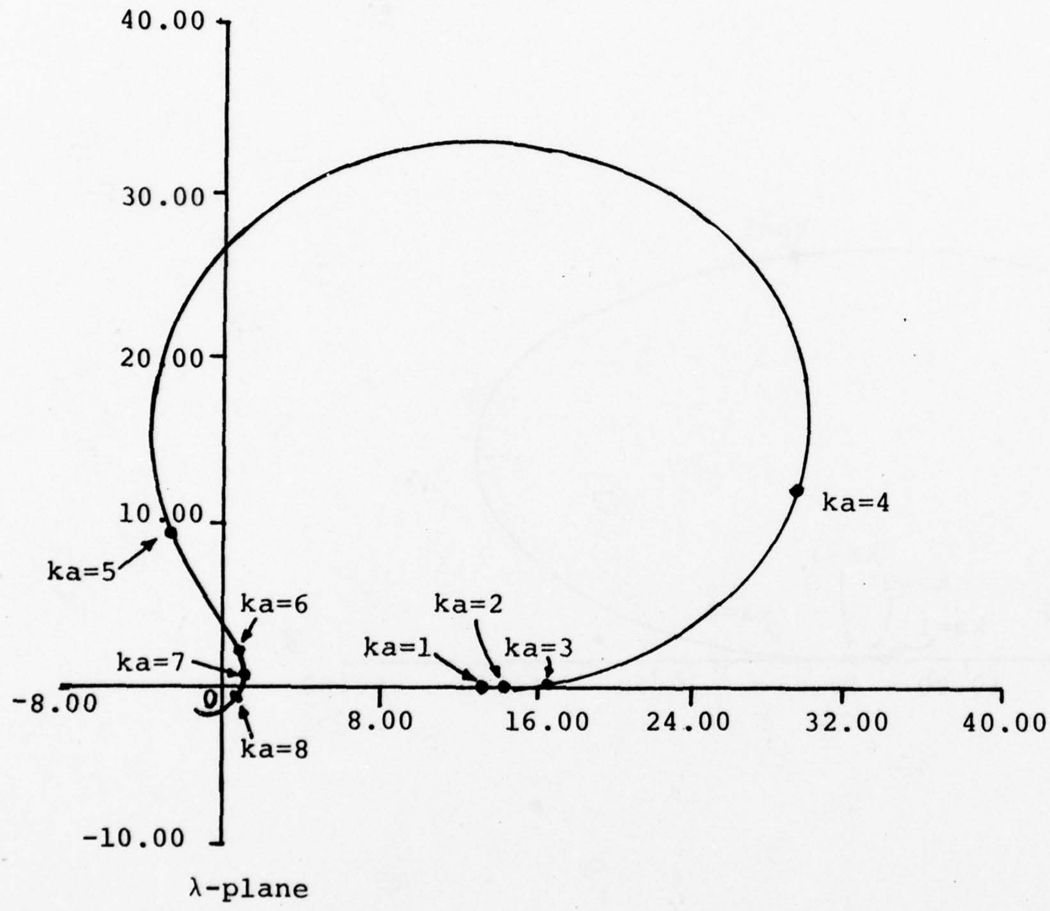


Figure 7. $\lambda_n(ka)$, $n = 6$

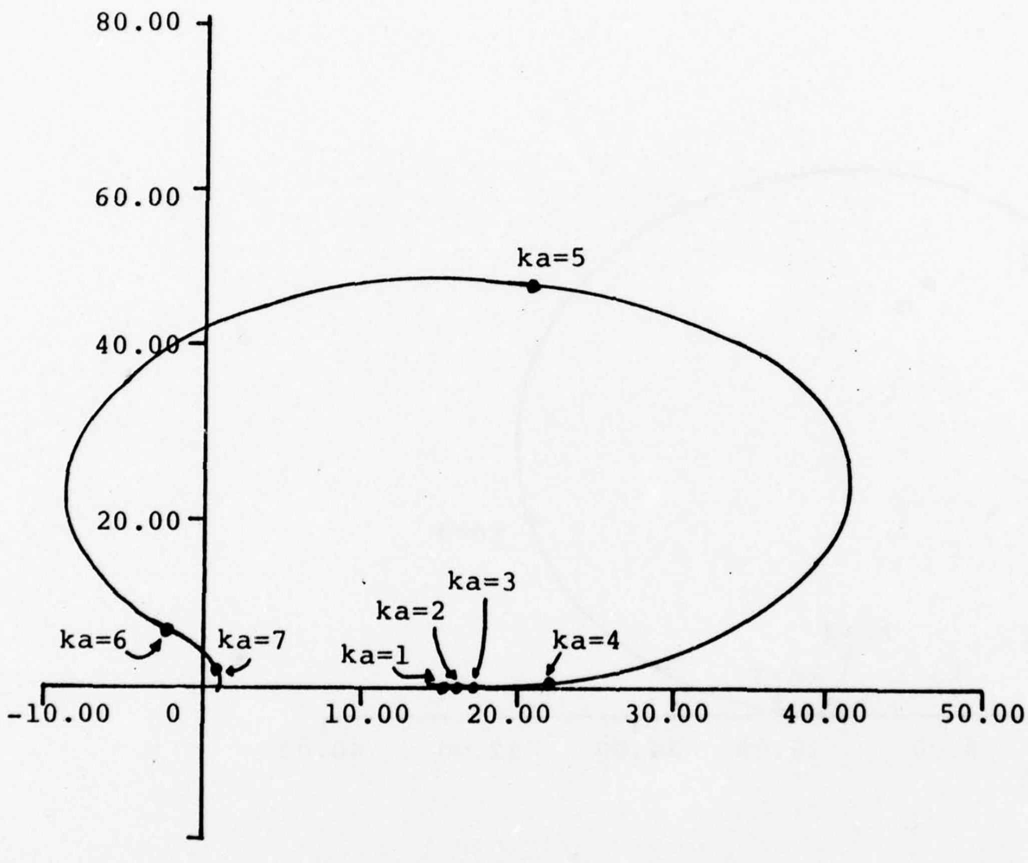


Figure 8. $\lambda_n(ka)$, n = 7

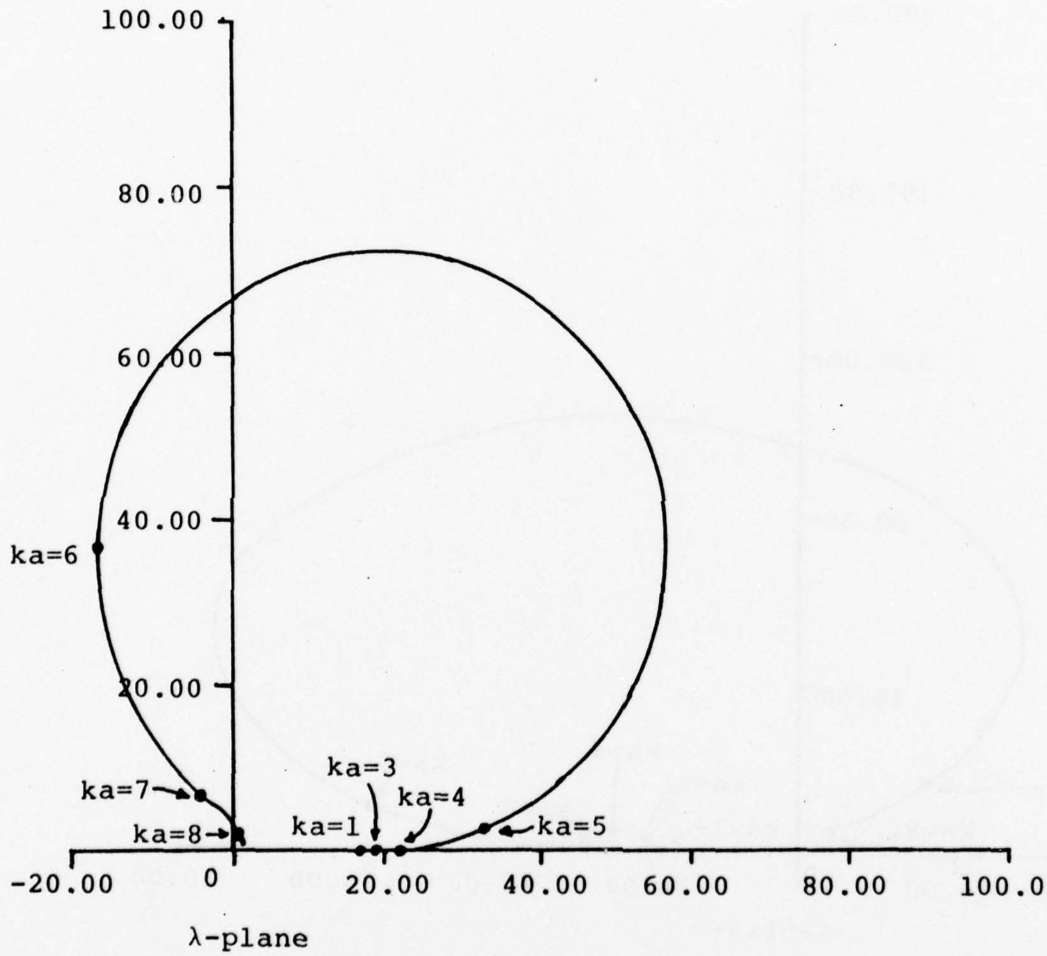


Figure 9, $\lambda_n(ka)$, n = 8

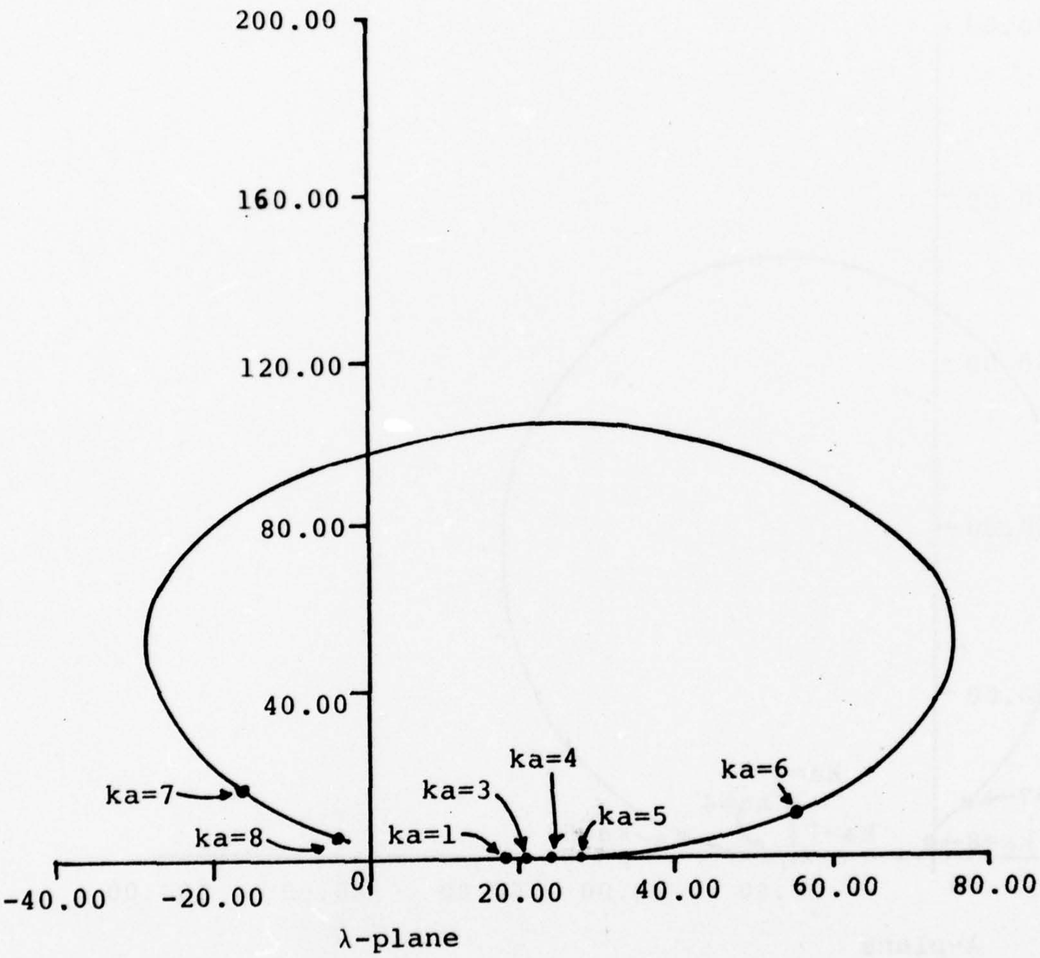


Figure 10. $\lambda_n(ka)$, $n = 9$

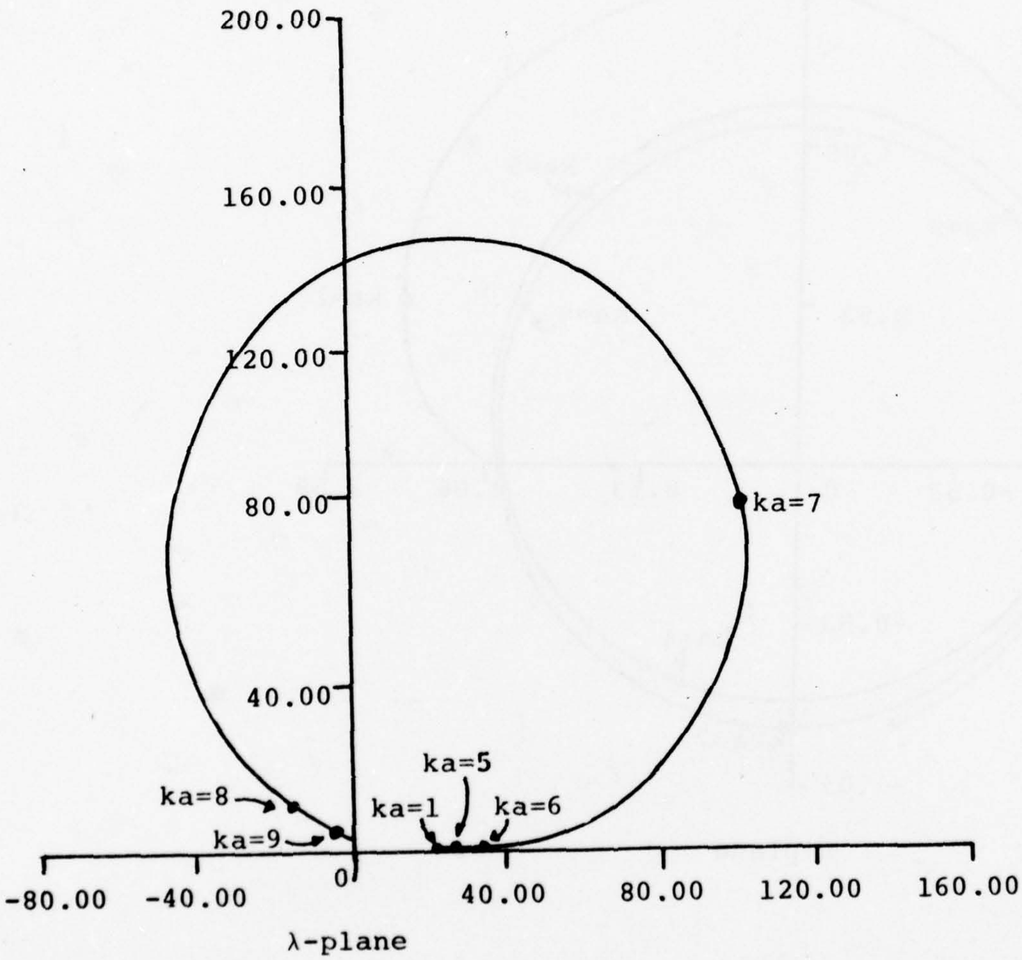


Figure 11. $\lambda_n(ka)$, $n = 10$

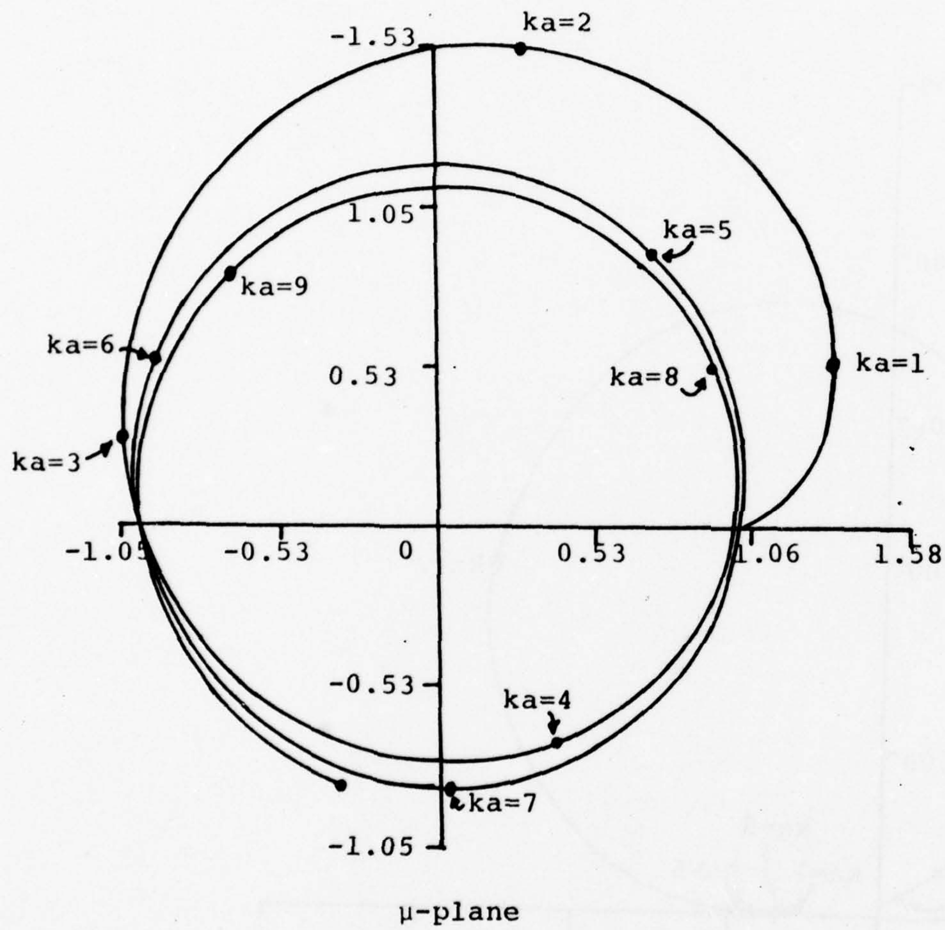


Figure 12. $\mu_n(ka) = \frac{1}{\lambda_n}$, n = 0

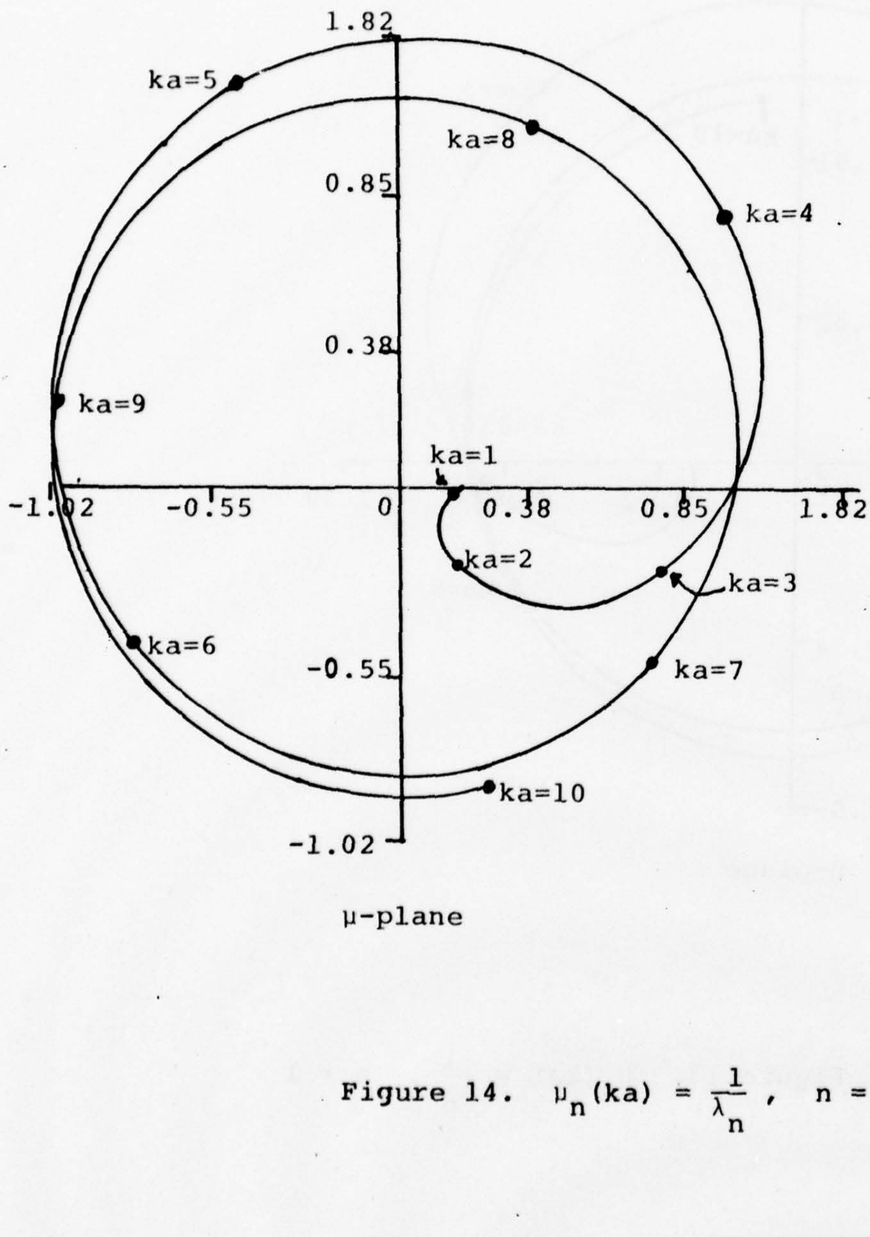


Figure 14. $\mu_n(ka) = \frac{1}{\lambda_n}$, $n = 2$

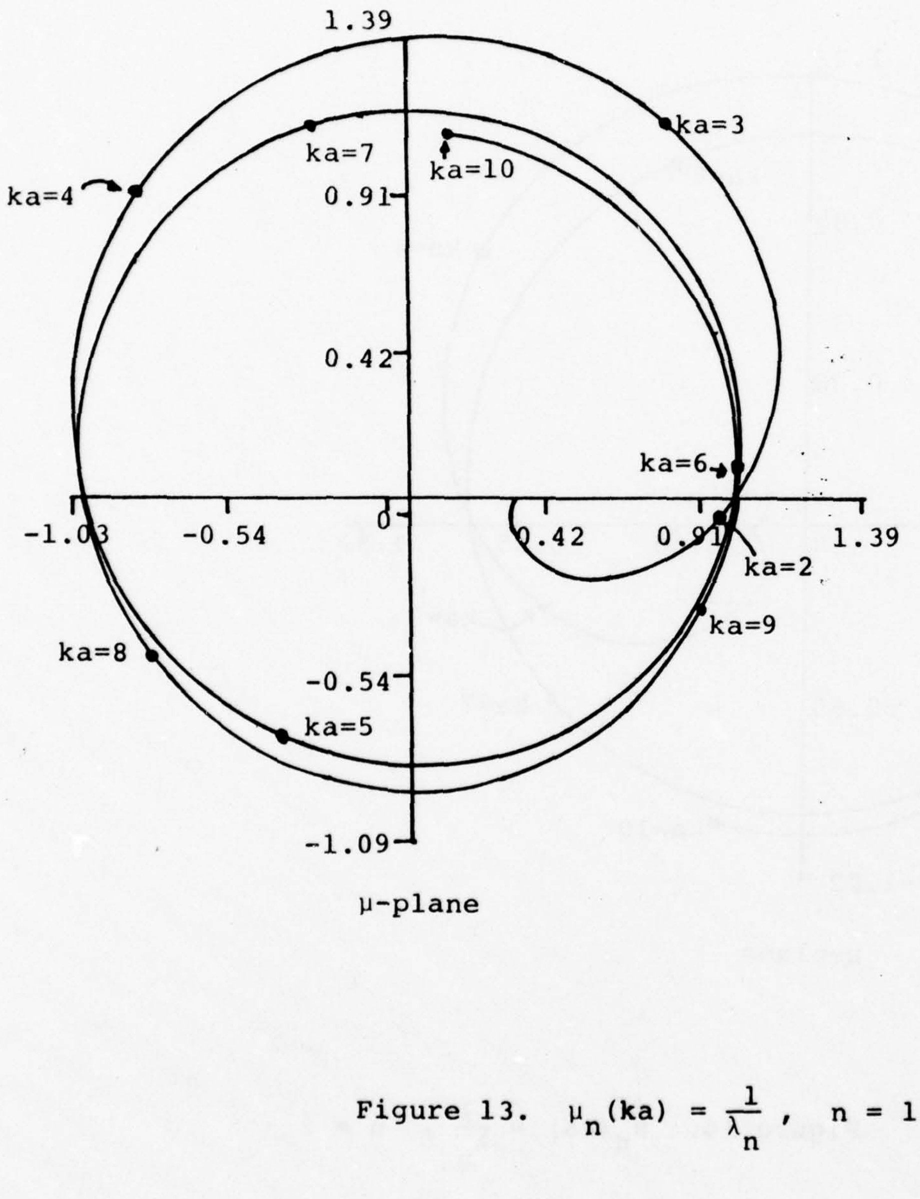


Figure 13. $\mu_n(ka) = \frac{1}{\lambda_n}$, $n = 1$

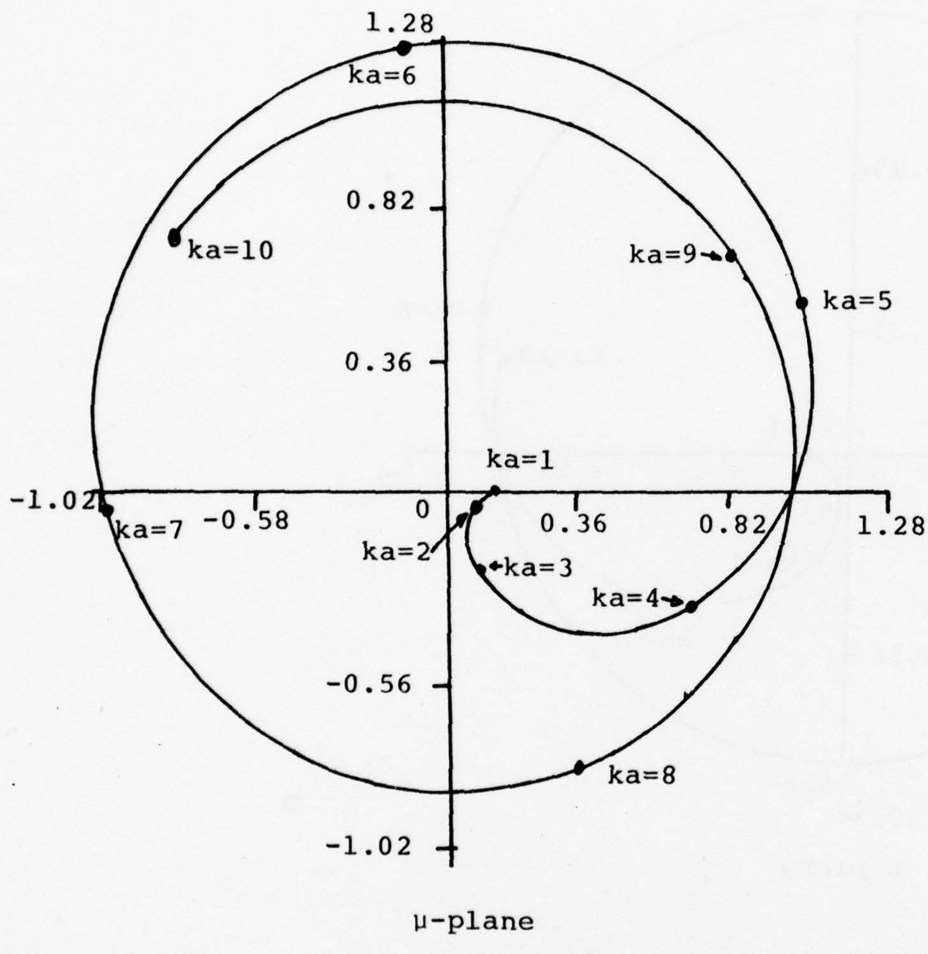


Figure 15. $\mu_n(ka) = \frac{1}{\lambda_n}$, $n = 3$

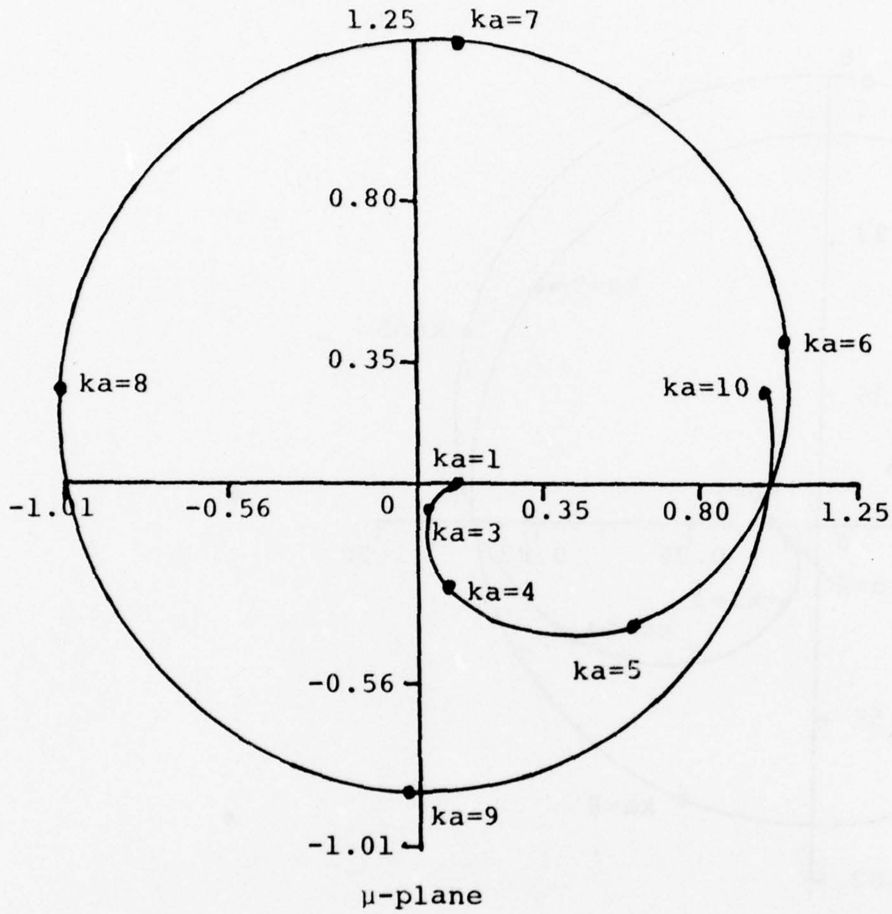


Figure 16. $\mu_n(ka) = \frac{1}{\lambda_n}$, $n = 4$

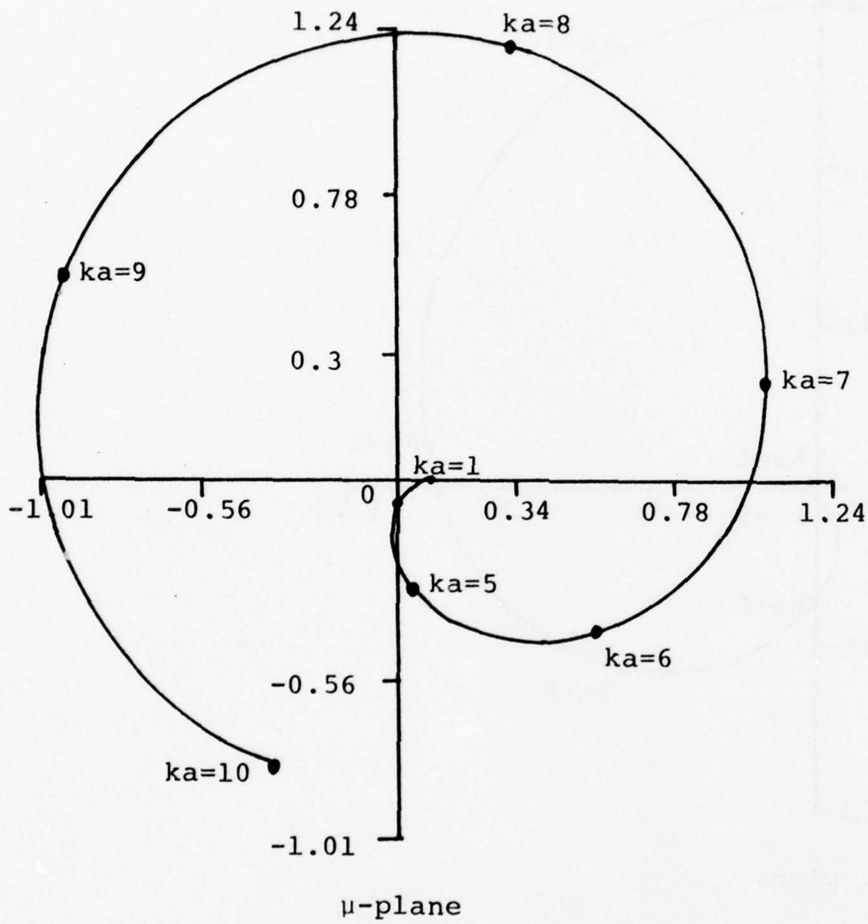


Figure 17. $\mu_n(ka) = \frac{1}{\lambda_n}$, n = 5

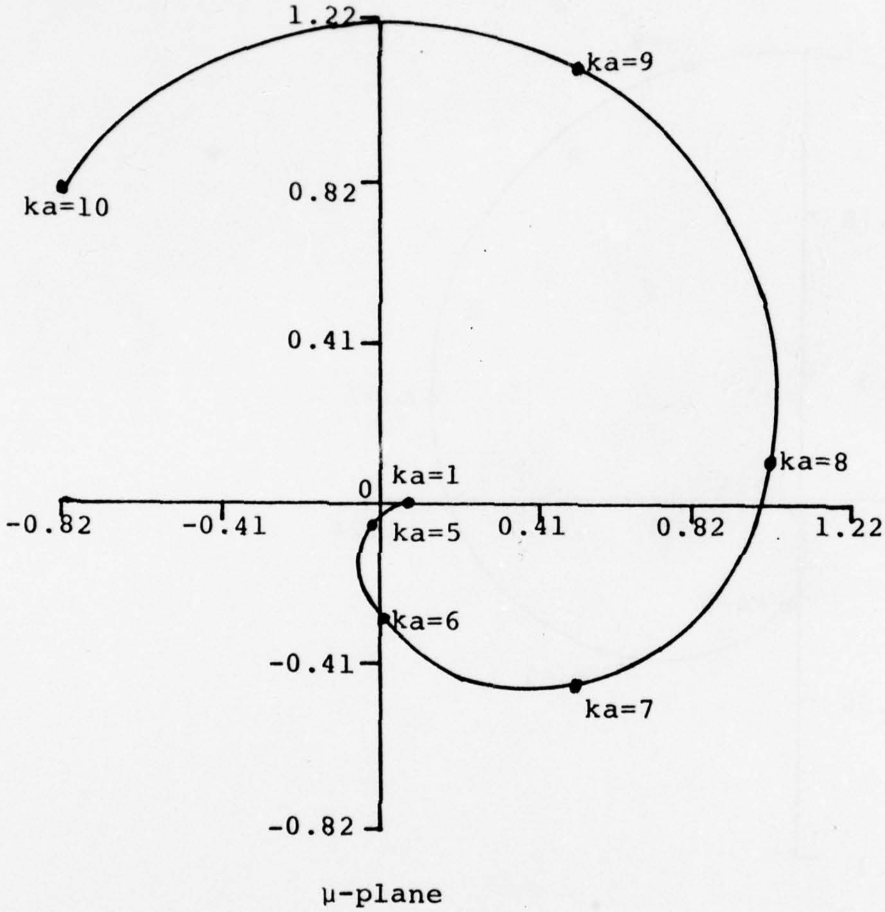


Figure 18. $\mu_n(ka) = \frac{1}{\lambda_n}$, $n = 6$

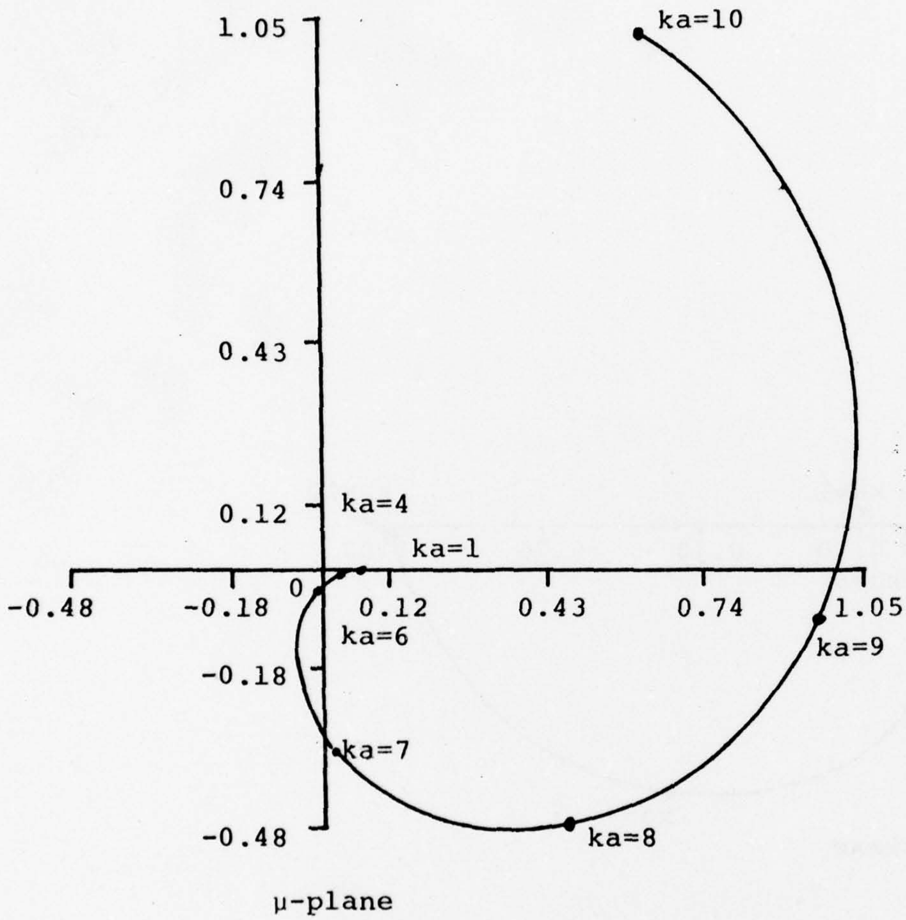


Figure 19. $\mu_n(ka) = \frac{1}{\lambda_n}$, $n = 7$

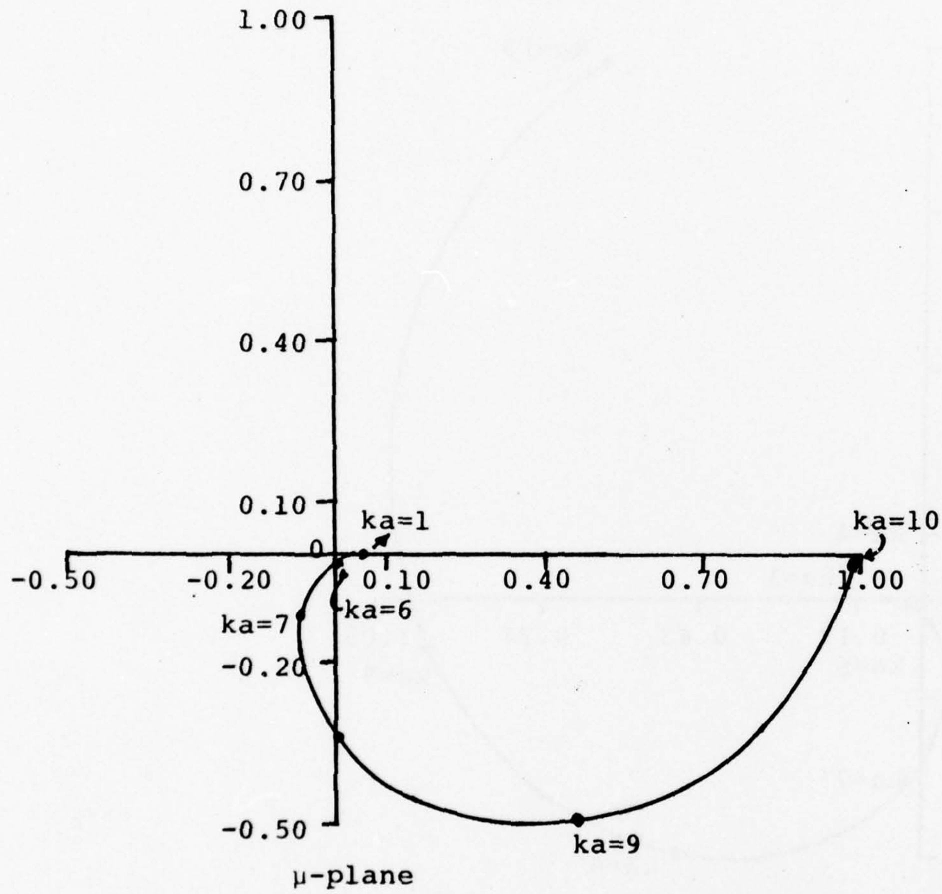


Figure 20. $\mu_n(ka) = \frac{1}{\lambda_n}$, n = 8

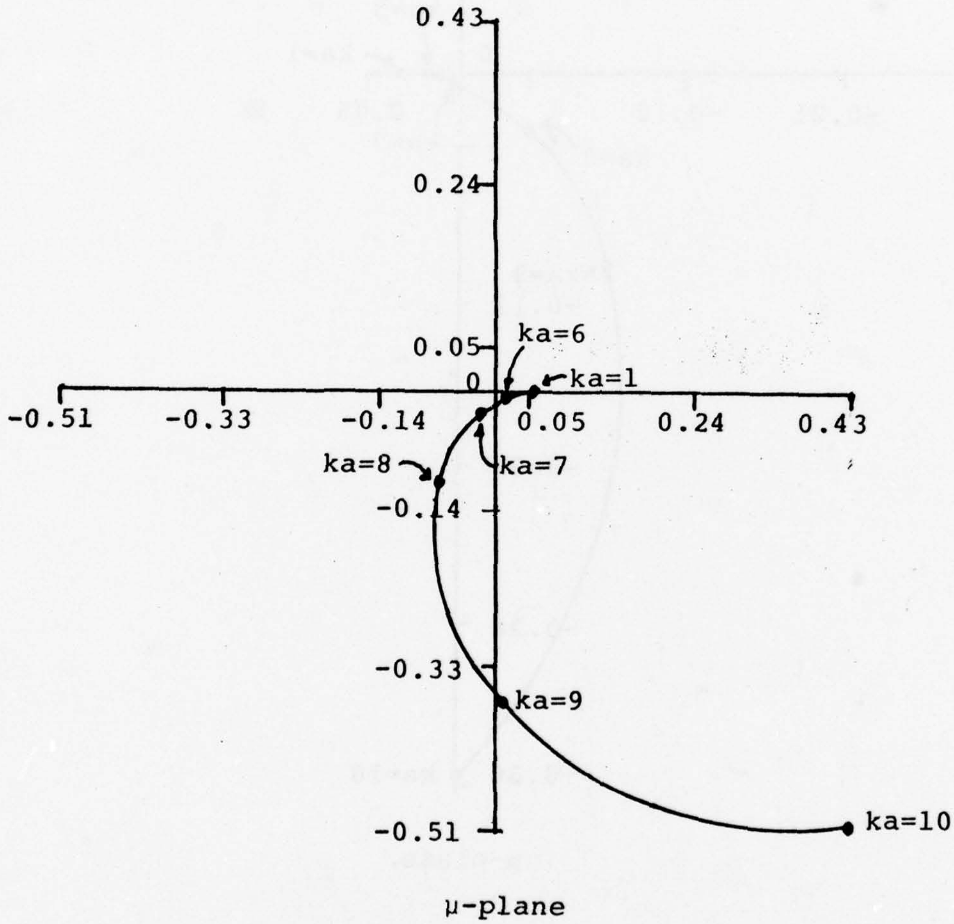


Figure 12. $\mu_n(ka) = \frac{1}{\lambda_n}$, n = 9

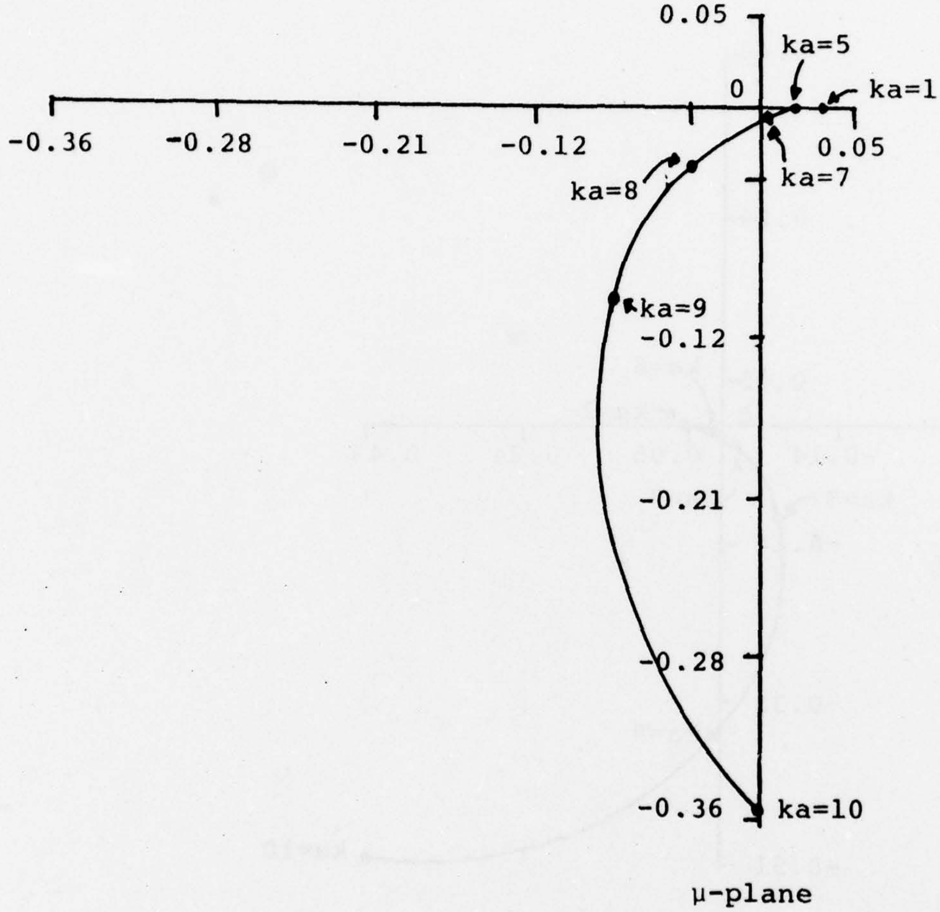


Figure 22. $\mu_n(ka) = \frac{1}{\lambda_n}$, $n = 10$

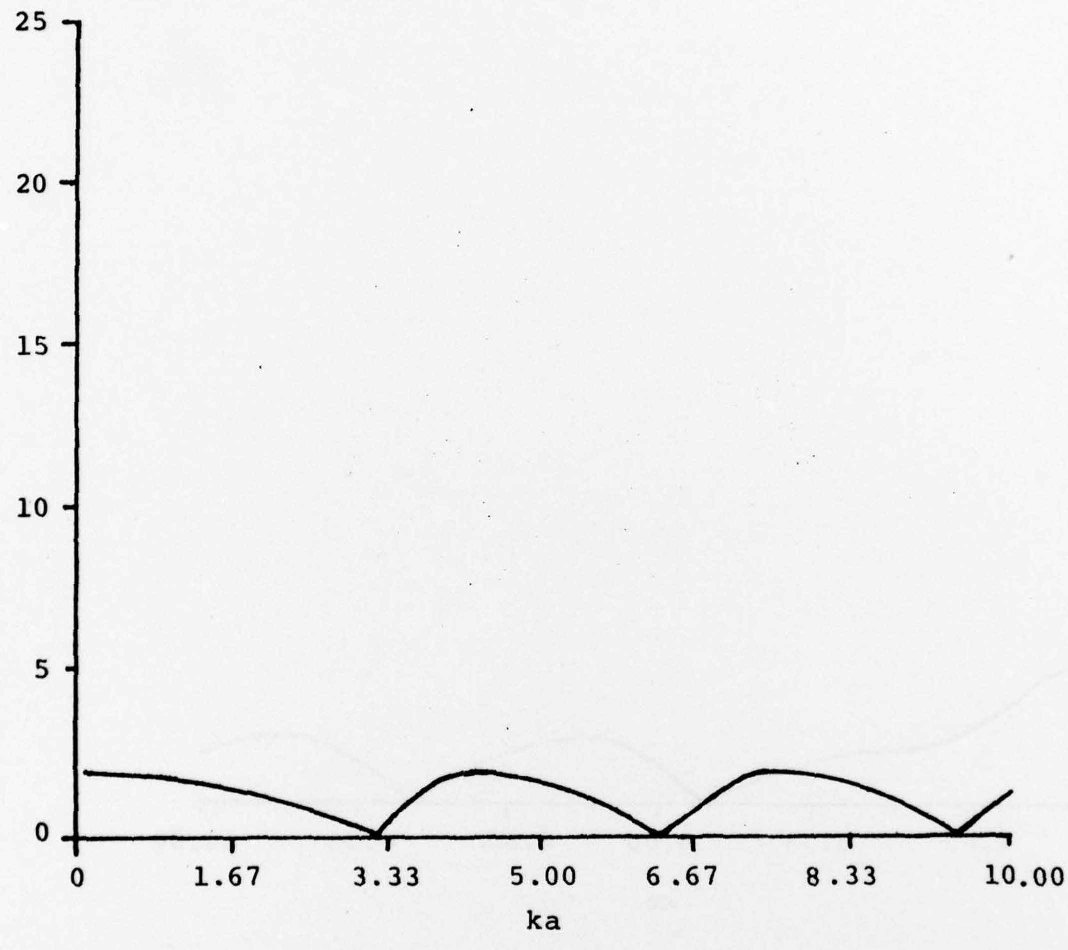


Figure 23. $|\lambda_n(ka)-1|$, $n = 0$

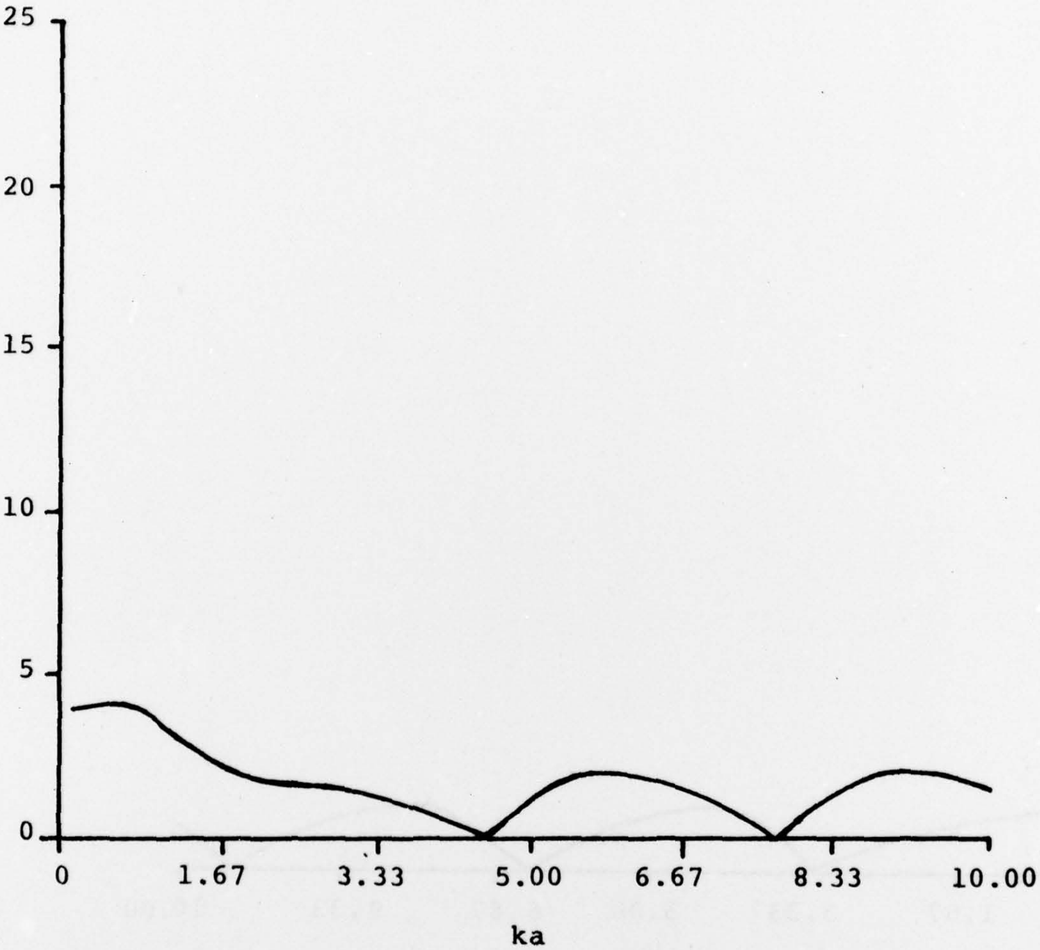


Figure 24. $|\lambda_n(ka)-1|$, n = 1

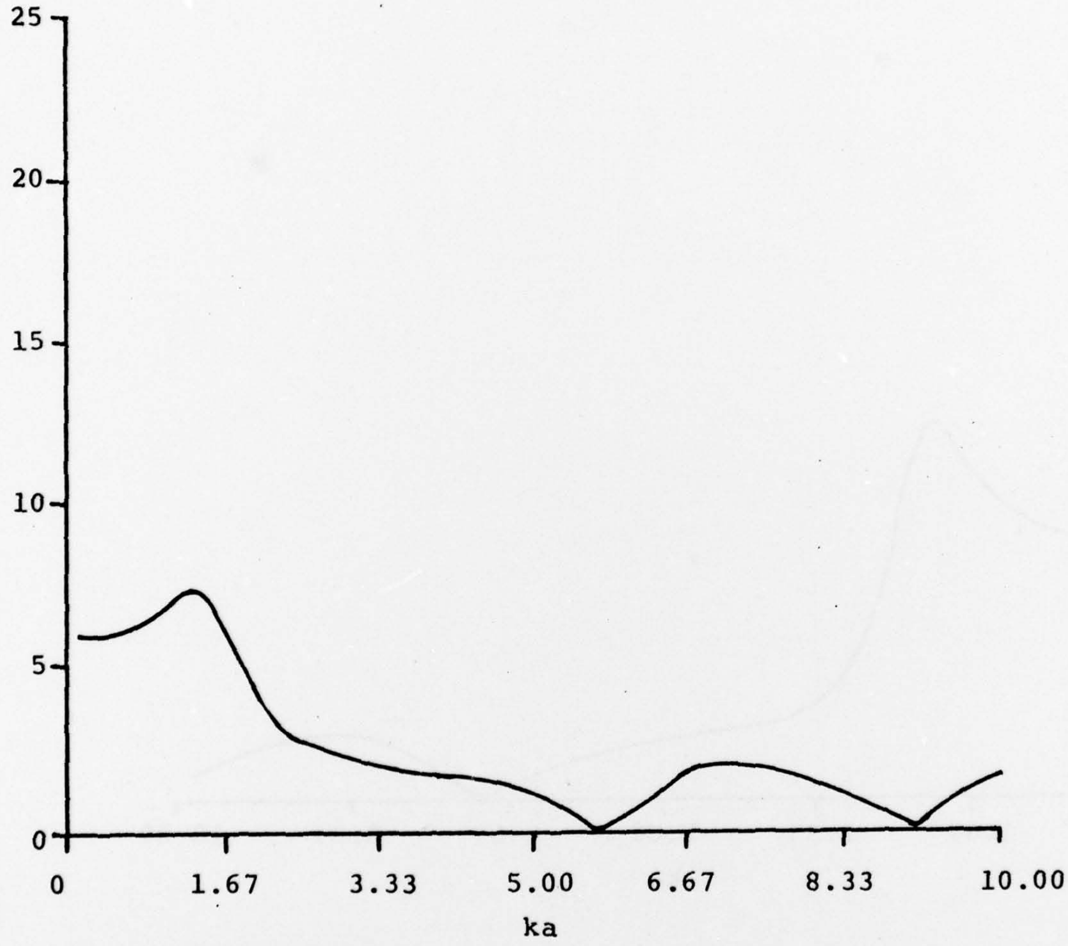


Figure 25. $|\lambda_n(ka)-1|$, n = 2

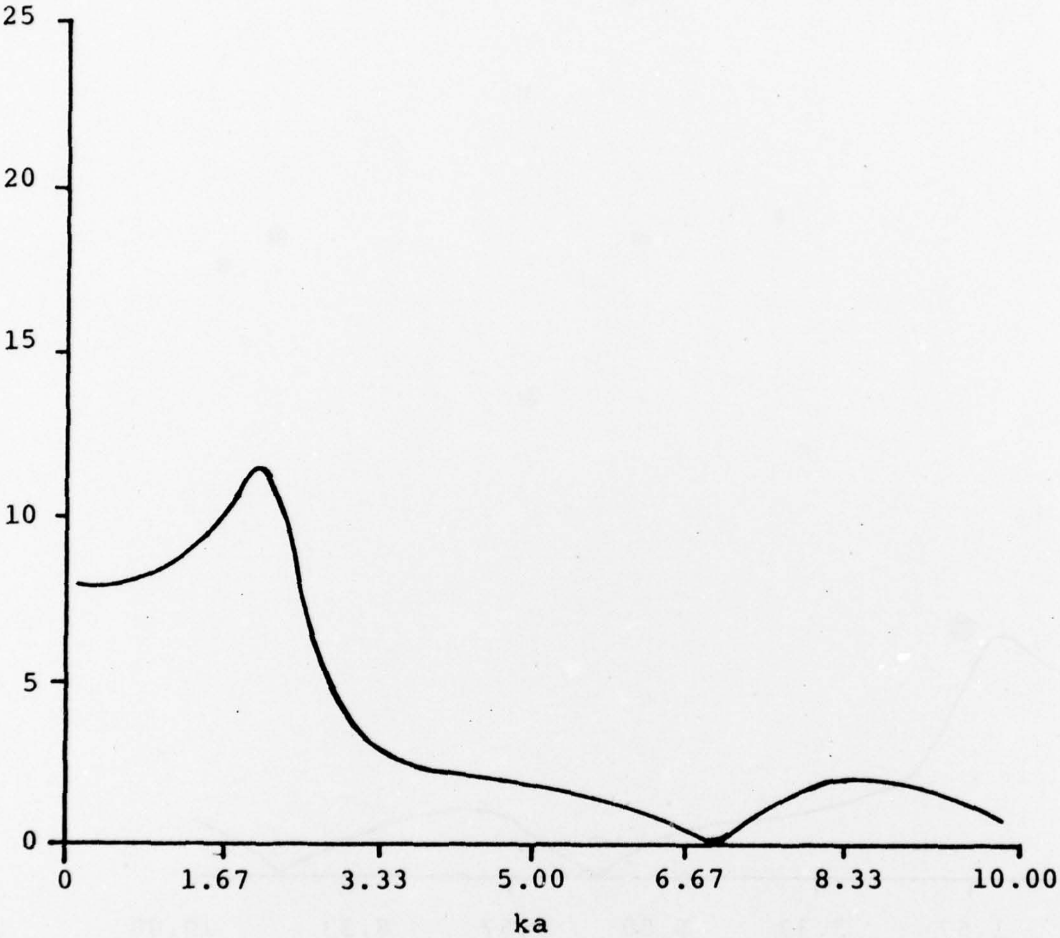


Figure 26. $|\lambda_n(ka)-1|$, $n = 3$

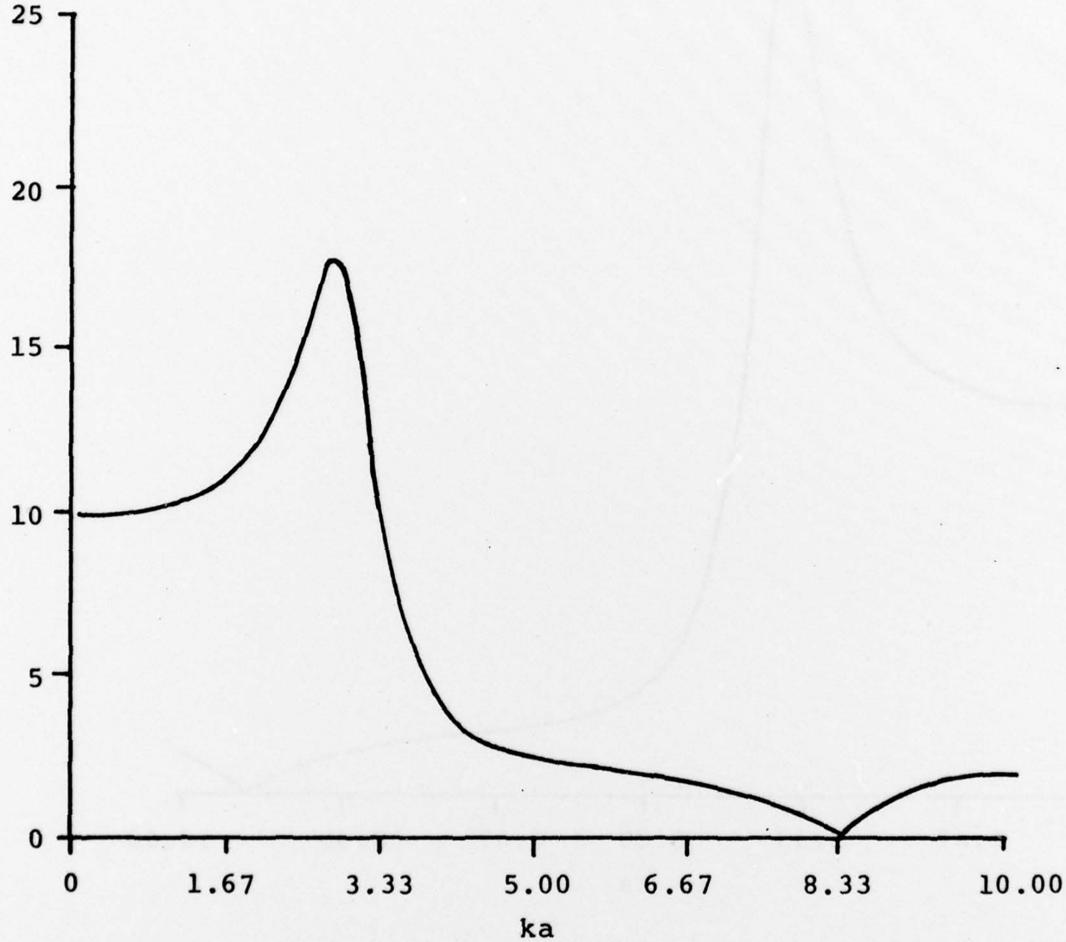


Figure 27. $|\lambda_n(ka)-1|$, n = 4

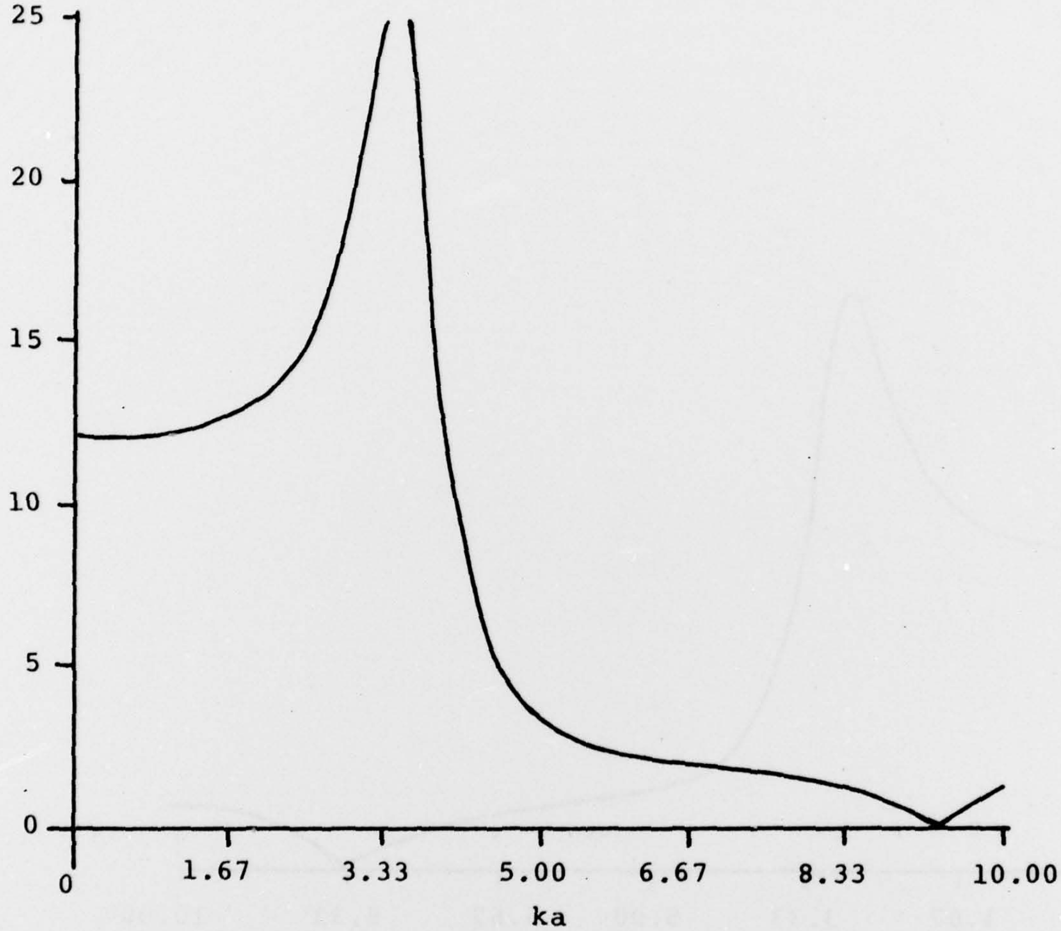


Figure 28. $|\lambda_n(ka)-1|$, n = 5

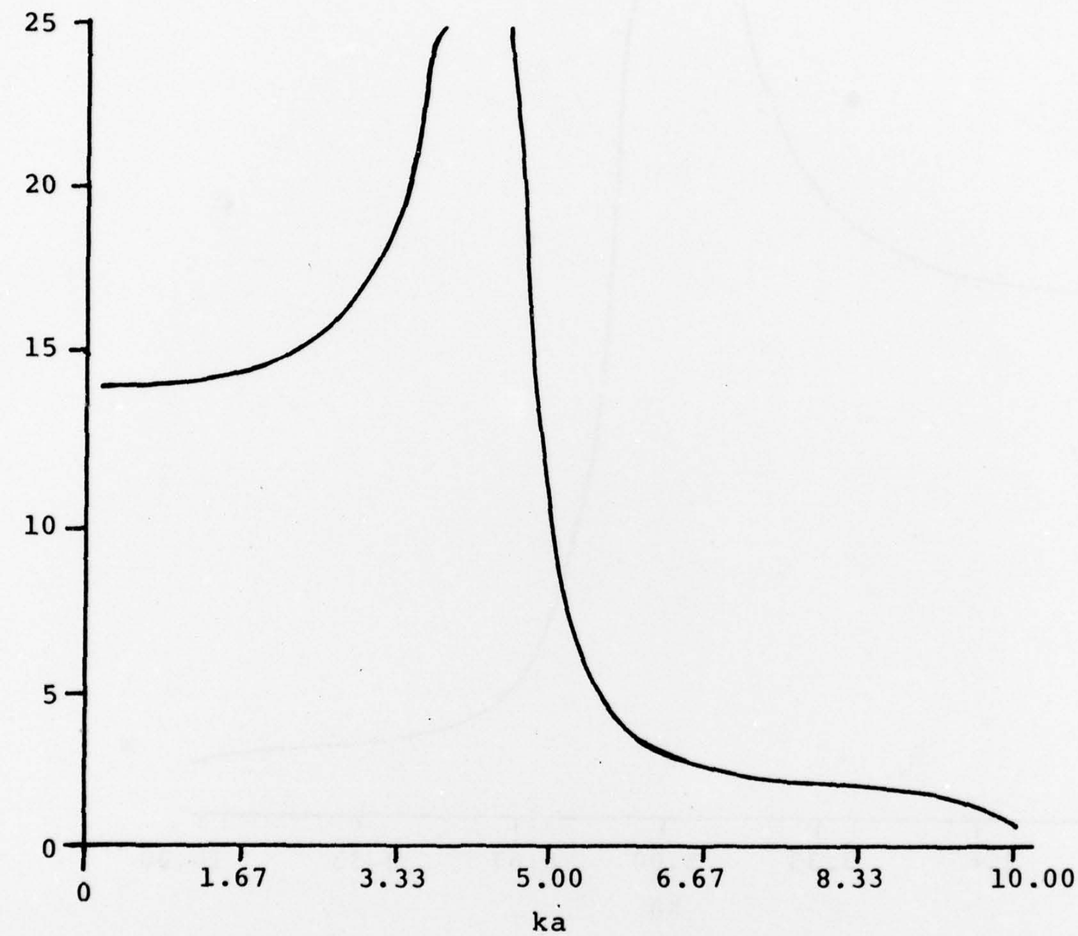


Figure 29. $|\lambda_n(ka)-1|$, n = 6

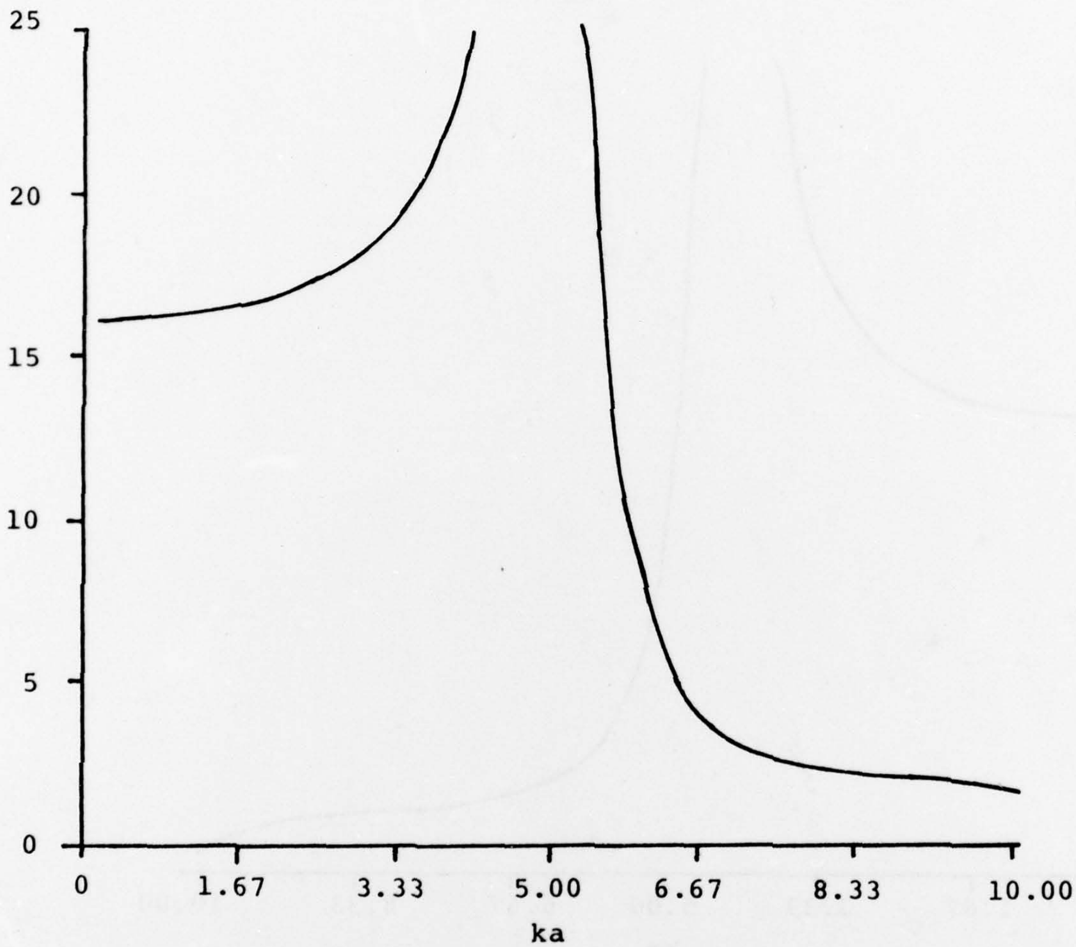


Figure 30. $|\lambda_n(ka)-1|$, n = 7

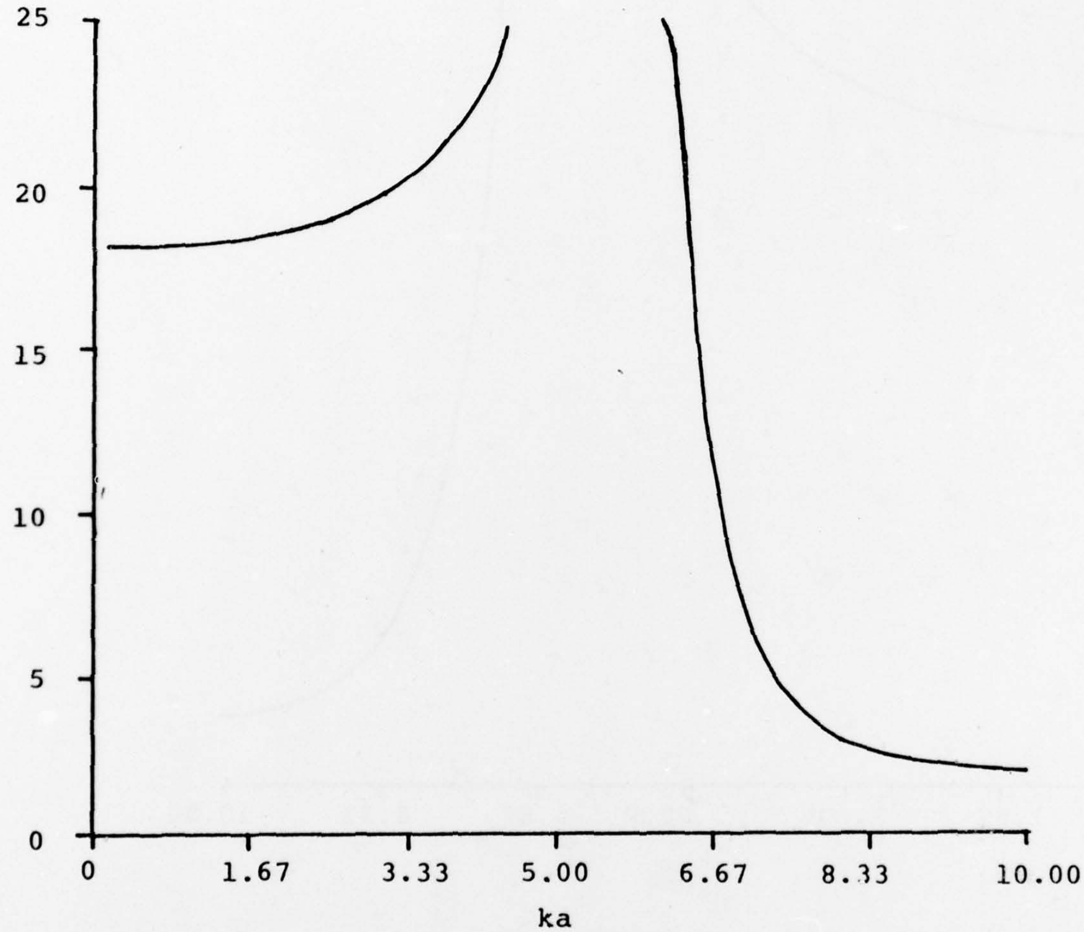


Figure 31. $|\lambda_n(ka)-1|$, n = 8

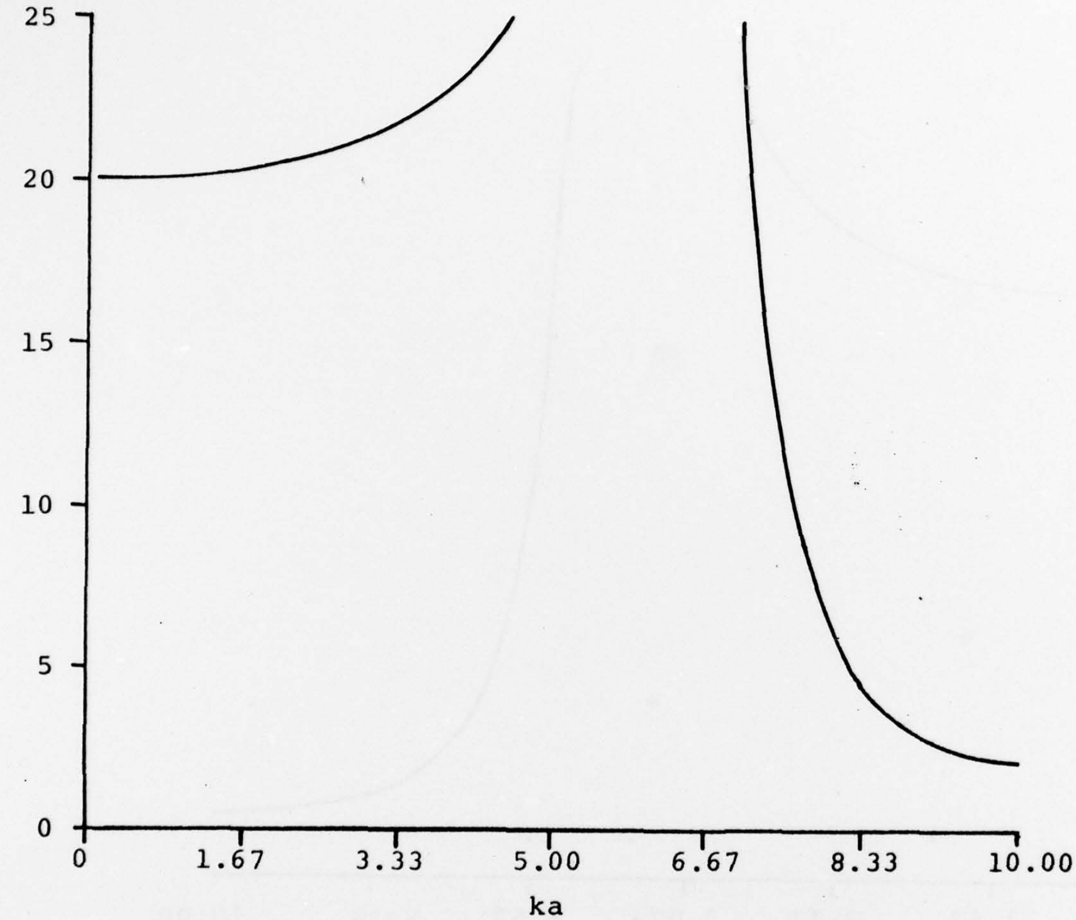


Figure 32. $|\lambda_n(ka)-1|$, n = 9

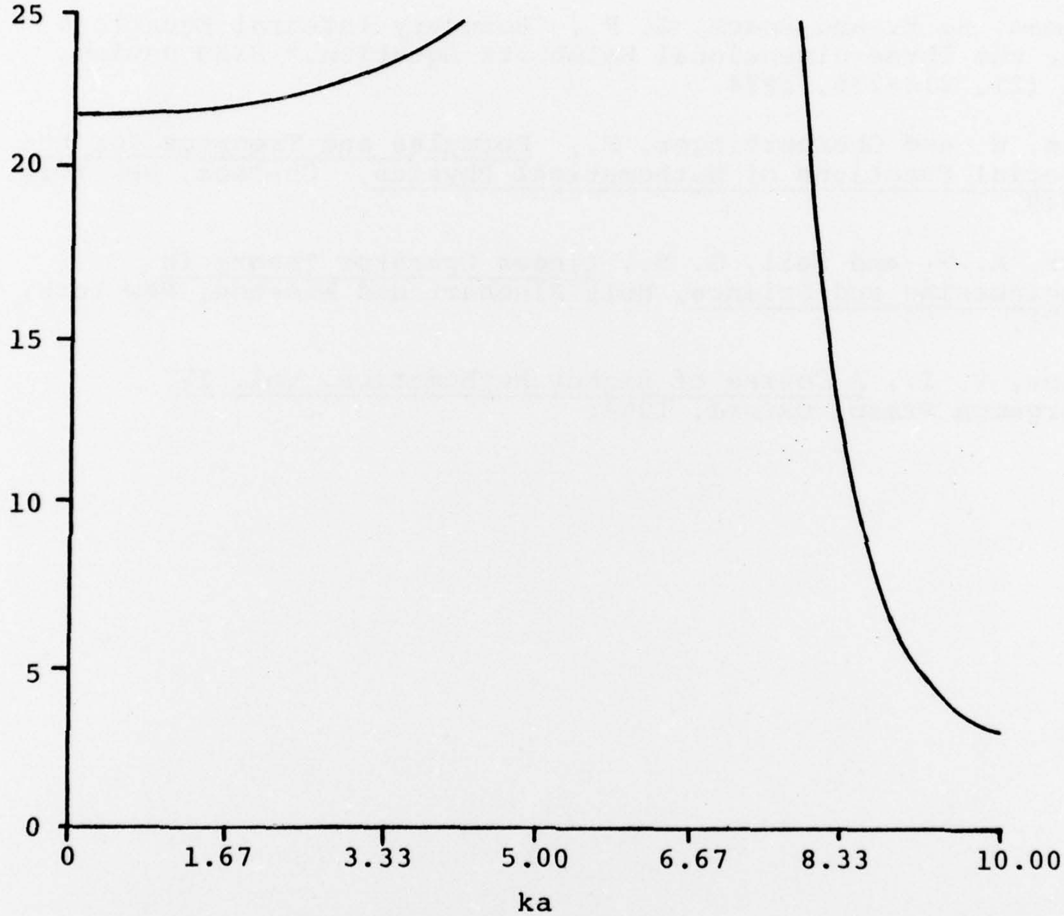


Figure 33. $|\lambda_n(ka)-1|$, $n = 10$

<u>References.</u>

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19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

scattering; integral operators; operator valued functions; parameter dependent eigenvalues

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The boundary integral operator which arises in a double layer formulation of the Neumann problem for the Helmholtz equation is analyzed as an operator valued function of wave number in the particular case of a spherical boundary. The spectrum of the operator is found and its explicit dependence on wave number is exhibited, both analytically and numerically. In addition, the explicit polar decomposition of the operator is carried out and it is shown that

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20. Abstract continued.

asymptotically the operator becomes selfadjoint for small values of wave numbers and unitary for large values. Advantages of operator factorization are discussed.